AN ELEMENTARY TREATISE

ON

KINEMATICS AND DYNAMICS

BY

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PREFACE.

A NEW edition of this book, with but few changes, seems to be justified by the fact that it has been found useful by teachers in its original form.

It is an elementary deductive treatment of what is commonly called Abstract Dynamics, with such portions of Concrete or Physical Dynamics as are required to furnish illustrative material. The student is recommended to make an inductive study of some parts of the subject in the laboratory before reading it.

The book is designed for use in the higher classes of schools and the junior classes of colleges, and assumes, consequently, a knowledge of those portions only of mathematics which are usually taught in schools. Unfortunately the mathematical range of school work does not include the elements of the Calculus ; and as rates of change must be dealt with in any serious study of Dynamics, the Calculus has had to be "dodged," in most cases doubtless in a laborious manner.

The distinctive characteristic of the book is its comprehensiveness. It is not a selection of those sections of Dynamics which find favour with examining bodies, but a systematic treatment of the whole subject. And the endeavour has been made so to analyze the subject that

the reader may be able to recognize readily the relation which each department bears to the whole.

Kinematics has consequently been treated by itself. But the teacher who prefers his pupils to study the corresponding kinematical and dynamical problems *pari passu*, will find that he can readily direct their reading to this end.

The treatment is based upon the ordinary conception of force, because actual experience always involves this conception, and any other treatment, however logical, requires an abstraction from experience which it seems undesirable to ask the student to make at an early stage.

In discussing the dynamics of translation and rotation, it has been assumed that bodies may be regarded as consisting of particles exerting forces on one another at a distance; and in discussing the dynamics of deformation, that they may be regarded as consisting of elements exerting forces on contiguous elements only. The only test of the philosophical defensibility of such hypotheses is their utility. The fundamental dynamical assumptions take different forms in the two cases.

The illustrative problems have been so selected as both to exercise the student in the application of his mathematical knowledge, and to enable him to test his grasp of principles by applying them to numerical examples. And these examples have been expressed in terms of various sets of units because it seems desirable that the student should acquire an equal mastery over all.

The fact that very few corrections have had to be made in the solutions of the examples is due to the careful revision of them in the first edition made by a

former colleague, Professor D. A. Murray, of Dalhousie College, Halifax, N.S., to whom I desire to express my thanks anew.

In the preparation of my class lectures, which formed the basis of this book, I derived assistance, more or less direct, from so many sources that it is best perhaps to make special mention of none. But I cannot refrain from making grateful reference to the lectures of my first teacher, the late Professor Charles Macdonald, of Dalhousie College, and to the lectures and published works of my illustrious predecessor, the late Professor P. G. Tait.

J. G. MACGREGOR.

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PART I.—KINEMATICS.

CHAPTER I.

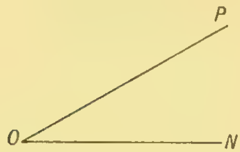
POSITION AND MOTION.

1. Kinematics is that branch of Mathematical Science which investigates motion. It makes no inquiry as to the causes of the changes of motion in bodies, but studies their motion in itself.

2. *Position.*—We recognize bodies as existing in space and having definite positions among one another. We recognize them as having positions, however, only by the aid of neighbouring bodies, and we describe the positions of their various points by reference to chosen points in neighbouring bodies. Position in space is thus a relative conception. "Absolute position" is a meaningless phrase.

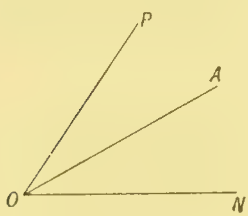
The position of a point P, relative to any other point O, is completely determined if we have sufficient data to enable us to proceed from O to P. There are various modes of specifying the necessary data. They are called systems of co-ordinates. Of these we may mention two: (1) that of Polar Co-ordinates; (2) that of Cartesian Co-ordinates.

3. *Polar Co-ordinates.*—If the point P is situated in a given plane, its position relative to O, another point in that plane, may be described by the aid of ON, a known line in the same plane, by a statement of the angle NOP and the length OP. Thus if O and P are points in a horizontal plane, and ON the north and south line through O, the angle NOP (which in that case is called the azimuth of P) and the distance of P from O determine the position of P.



The point O is called the pole in this system of co-ordinates, ON is called the initial line, and OP the radius vector. The length of OP and the magnitude of the angle NOP are the polar co-ordinates of the point P. They are usually denoted by the symbols r and θ.

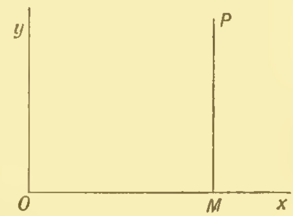
To describe the position of a point P not in a known plane, let O be the pole, ON the initial line, and ONA a known plane containing ON but not P. Let OA be the intersection of the plane ONA with a plane perpendicular to it through OP. Then, if the angles NOA and AOP are given, the direction of OP is known, and if the length of OP is also given, the position of P is completely determined. The length of OP and the angles NOA and AOP are then the polar co-ordinates of P. They are usually denoted by r, ϕ, and θ respectively.



O, for example, may be a point on the earth's surface, ONA the horizontal plane, ON the north and south line through O; in which case the angles NOA and AOP are what are called in astronomy the azimuth and altitude of P.

4. *Cartesian Co-ordinates.*—If the point whose position is to be specified is known to be situated in a given plane, its position may be described by a statement of the

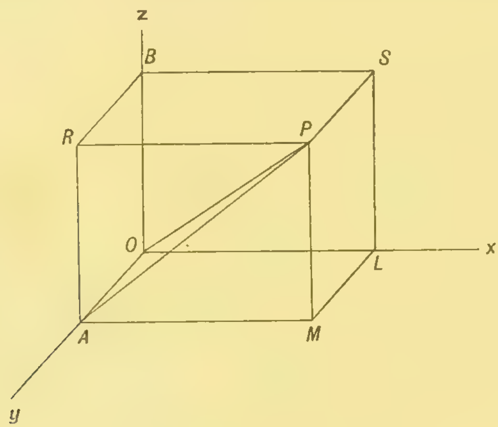
distances which must be traversed in directions parallel to two known lines in that plane, in passing from the point of reference to the given point. Let O be the point of reference, Ox and Oy two known lines in the given plane, and P the point whose position is to be described. From P draw PM parallel to Oy. If the lengths of OM and MP are given, the position of P is determined. These lengths are called the Cartesian co-ordinates of P (this system having been first employed by Descartes). The point O is called the origin of co-ordinates, and the lines Ox and Oy the axes of co-ordinates—the one the axis of x or the x-axis, the other the axis of y or the y-axis. If the axes are perpendicular to one another they are said to be rectangular, and the co-ordinates are called *rectangular co-ordinates*. In that case OM is called the *abscissa*, MP the *ordinate*. These co-ordinates are usually denoted by the symbols x and y respectively. A point whose co-ordinates are x and y is called the point (x, y).



If P is situated to the left of Oy, the distance OM must be traversed from O in the opposite direction to that shown in the diagram. In that case, the co-ordinate OM is considered negative. Similarly, if P is below Ox, the co-ordinate MP is considered negative. Thus a point to the left of Oy and above Ox will have the co-ordinates $-x, y$; one to the left of Oy and below Ox the co-ordinates $-x, -y$.

5. If the point P is not known to be in a given plane, its position may be described by reference to three axes drawn through the origin in known directions in space. Let Ox, Oy, Oz be three such axes. From P draw PM parallel to Oz and meeting the plane Oxy in M. From M draw ML parallel to Oy and meeting Ox in L. The position of P is specified if OL, LM, MP are given, *i.e.*,

if the distances are known which must be traversed in the directions of the axes, in passing from O to P. OL,



LM, MP, the co-ordinates of P, are usually denoted by the symbols x, y, z respectively.

If Ox, Oy, Oz are at right angles to one another, the co-ordinates are said to be rectangular. O, for example, being a point on the surface of the earth, Ox the north and south line, Oy the east and west line, and Oz the vertical line, the co-ordinates of P are the distances that must be traversed northwards, eastwards, and upwards in order to reach P.

The same convention as to signs is employed as in 4, co-ordinates drawn from O in directions opposite to those of Ox, Oy, Oz respectively being considered negative. With this convention no two points in space can have the same co-ordinates.

6. If from P lines be drawn parallel to OL and LM, and meeting the Oyz and Oxz planes in R and S respectively, and if from Oy and Oz, OA and OB be cut off equal to LM and MP respectively, and if MA, RA, RB, SB, SL be joined, it will be clear that OP is the diagonal of a paralellopiped, of which OL, LM, and MP are the

edges; and it follows that the same point P is reached in whatever order the distances OL, LM, MP may be traversed.

It will also be clear that, if the co-ordinates are rectangular, PM is the distance of P from the plane containing the axes of x and y, or, as it is called, the plane of xy; and that OL and ML are the distances of P from the planes of yz and of xz respectively. Hence the rectangular co-ordinates of a point may be taken to be its distances from three planes which intersect in three straight lines at right angles to one another. Thus the position of a point in a room is completely specified if its distances from the floor and from any two adjacent walls are given.

7. The direction of the line OP may be specified by giving the angles α, β, γ, at which it is inclined to the rectangular axes of x, y, z respectively. In that case one additional datum will complete the specification of the position of P, viz., the length of OP. In this mode of specifying the position of a point four data appear at first sight to be required. But as AP is perpendicular to OA,

$$OA \text{ or } y = OP \cos\beta.$$

Similarly
$$x = OP \cos\alpha,$$
$$z = OP \cos\gamma.$$

Now OP being the diagonal of a rectangular parallelopiped whose edges are x, y, z,

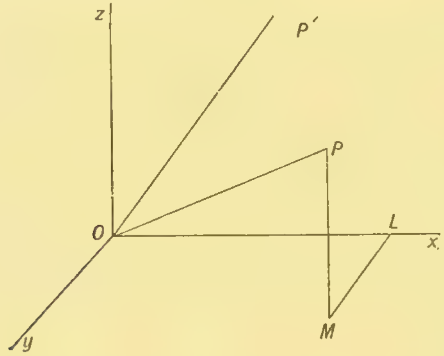
$$OP^2 = x^2 + y^2 + z^2.$$

Hence
$$OP^2 = OP^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. There is therefore always this relation between three of the four data.

As α, β, γ determine the direction of OP, the cosines of these angles are called the *direction cosines* of OP.

8. It is frequently convenient to be able to express the inclination* of two straight lines in terms of their direction cosines. Let OP and OP' be two such lines or



lines drawn parallel to them, Ox, Oy, Oz rectangular axes, OL, LM, MP the rectangular co-ordinates of any point P of OP, and α, β, γ the angles of inclination of OP to the axes of x, y, and z respectively. Then

$$OL = OP \cos \alpha, \quad LM = OP \cos \beta, \quad MP = OP \cos \gamma.$$

Now the projection† of OP on OP' is equal to the sum of

*The inclination of one straight line to another, whether they are in one plane or not, is the angle between two lines drawn parallel to them from any point.

†(1) The foot of the perpendicular from a point on a straight line is called the orthogonal projection or simply the projection of the point on the line.

(2) The locus of the projections of all the points of any line on a given straight line is called the projection of the former on the latter.

(3) The projection of a finite straight line on a straight line is equal in length to the product of the length of the projected line into the cosine of its inclination to the given straight line. Let LM be the projected line, AB the line on which it is projected. In general these lines will not be in the same plane. From L, M, draw Ll, Mm, perpendicular to AB. Then lm is the projection of LM. From m draw ml' equal and parallel to ML, and join Ll' and ll'. Then Ll' is parallel to Mm and therefore perpendicular to AB.

the projections of OL, LM, and MP on the same line. Hence, if θ is the angle between OP and OP', and α', β', γ', the inclinations of OP' to the axes of x, y, and z respectively,

$$OP\cos\theta = OL\cos\alpha' + LM\cos\beta' + MP\cos\gamma'$$
$$= OP\cos\alpha\cos\alpha' + OP\cos\beta\cos\beta' + OP\cos\gamma\cos\gamma'.$$

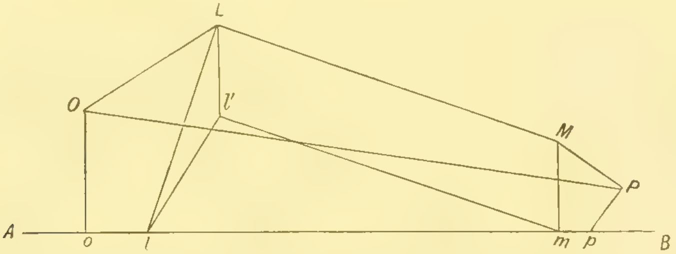
Hence

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

9. To find the value of $\sin\theta$, call $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, l, m, and n respectively, and $\cos\alpha'$, $\cos\beta'$, and

Hence the plane Ll' and therefore the line ll' are perpendicular to AB. Hence $\quad lm = l'm\cos lml' = LM\cos lml'$,

i.e., the projection of LM is equal to the product of LM into the cosine of its inclination to AB. The simpler case in which LM and AB are in one plane may be left to the reader.



(4) The algebraic sum of the projections of the parts of a broken line is equal to the projection of the straight line joining its end points. Let $OLMP$ be a broken line, the straight portions of which, OL, LM, MP, are not in one plane. From O and P draw Oo and Pp perpendicular to AB. Then ol, lm, and mp are the projections of OL, LM, and MP on AB. Also, from the construction, op is the projection of the line OP on AB. And $op = ol + lm + mp$. Hence the projection of OP on AB is equal to the sum of the projections of OL, LM, and MP on the same line.

If the position of L is such that the point l is situated to the left of o, ol being drawn to the left instead of the right must be considered negative, the lines lm and mp being taken as positive. In that case we have $op = lm + mp - ol$, *i.e.*, the projection of OP on AB is equal to the algebraic sum of the projections of OL, LM, and MP on the same line.

$\cos\gamma'$, l', m', and n' respectively. Then

$$\sin\theta = \{1-(ll'+mm'+nn')^2\}^{\frac{1}{2}}$$
$$= \{(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2\}^{\frac{1}{2}}$$
$$= \{(mn'-nm')^2+(nl'-ln')^2+(lm'-ml')^2\}^{\frac{1}{2}}.$$

10. It is frequently convenient also to be able to express the direction cosines (λ, μ, ν) of the common perpendicular to two lines, in terms of the direction cosines (l, m, n, and l', m', n') of the lines themselves. For this purpose we have (8), since $\cos(\pi/2)=0$,

$$l\lambda+m\mu+n\nu=0,$$
$$l'\lambda+m'\mu+n'\nu=0.$$

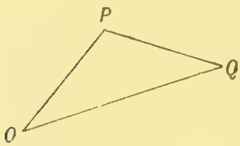
We have also (7) $\qquad \lambda^2+\mu^2+\nu^2=1.$

From these equations we obtain values of λ, μ, ν. Writing $\sin\theta$ for its value as given above (9) they are

$$\lambda=\frac{mn'-nm'}{\sin\theta};\quad \mu=\frac{nl'-ln'}{\sin\theta};\quad \nu=\frac{lm'-ml'}{\sin\theta}.$$

11. The positions of any two points relative to a third being given, that of either of the two relative to the other can be determined.

The positions of P and Q being given relatively to O, the lengths and directions of OP and OQ are known. Hence also (8) the angle POQ is known, and consequently all the sides and angles of the triangle OPQ. The direction and length of PQ being thus determined, the position of either of the two P, Q, relative to the other is known.



It follows that, if the positions of all the points of a system relative to any one are known, their positions relative to any other may be determined.

12. *Configuration.*—The arrangement of the points of a system is called its configuration. The configuration

of a system is thus known if the positions of all points relative to any one are known.

13. *Dimensions of Space.*—Whatever system of co-ordinates we may adopt, we require, in order to specify the position of a point, to have three quantities given. In the case of rectangular co-ordinates they are distances; in that of polar co-ordinates they consist of two angles and a distance. For this reason space is said to be tri-dimensional.

Similarly, any point in a given surface may be specified by a statement of two quantities, two distances or a distance and an angle; and any point in a given line may be specified by the statement of a distance merely. Hence a surface is said to have two dimensions, and a line one dimension.

14. *Measurement.*—The specification of the position of a point requires therefore that we should be able to measure lengths and angles.

The measurement of any quantity is the comparison of its magnitude with the magnitude of a known quantity of the same kind. The known quantity of the same kind is called *a standard or unit*; and a description of any measurement must include a statement of (1) the unit employed, and (2) the ratio of the magnitude of the quantity to be measured to the magnitude of the unit. This ratio is called the *numerical measure* or *value* of the quantity. Prof. James Thomson has proposed to shorten these terms to *numeric*.

Any quantity whatever, of the same kind as that to be measured, may be chosen as a standard or unit. But it will be evident that no standard should be employed which is not (1) constant in magnitude, (2) well known, and (3) easily reproduced; and we shall see farther on,

that among standards satisfying these conditions, there are reasons for preferring some to others.

15. We have seen that the numerical measure of any quantity in terms of any unit is the ratio of the magnitude of the quantity to that of the unit. It follows that the numerical measure of a given quantity must be inversely proportional to the magnitude of the unit in terms of which the quantity is expressed. Let Q be the numerical value of any quantity, and let $[Q]$ denote the magnitude of the unit in terms of which it is expressed. Then we have

$$Q \propto 1/[Q].$$

16. *Measurement of Length.*—The selection of standards of length presents no difficulty. A certain distance in space cannot, it is true, be marked off and kept; but a body, say a rod, may be selected and carefully preserved, and when it is in a specified physical condition (as to temperature, etc.), its length may be taken as unit of length. The submultiples of the unit thus chosen may then be determined by geometrical methods. For the various methods of comparing the length of a body or the distance between two points in space, with the standard length, the reader is referred to works on Laboratory Practice.

Different nations have adopted different units of length. The more important are the English and French units. The English unit, the *yard*, is defined by Act of Parliament to be the distance between the centres of two gold plugs in a certain bronze bar deposited in the Office of the Exchequer in London, the bar having the temperature 62°F. (The specification of the temperature is necessary, because the lengths of bodies vary with temperature.) The *foot* is one-third of the yard. The *inch* is one-twelfth of the foot. The *statute mile* is 1,760 yards. The French unit, the *mètre*, is the distance between the

end planes of a certain platinum bar deposited in Paris, the temperature of the bar being 0° C. The metre was intended to be the ten-millionth part of a quadrantal arc of a meridian on the earth's surface. It is now known to be a somewhat smaller fraction. The *decimetre*, *centimetre*, and *millimetre* are the tenth, hundredth, and thousandth parts of a metre respectively. The *decametre*, *hectometre*, and *kilometre* are equal to ten, one hundred, and one thousand metres respectively. The decimal division of the metre renders it a much more convenient unit than the yard.

The following table shows approximately the relative values of English and French units of length:

1 inch = 2·5400 cm.	1 centimetre = 0·39370 in.
1 foot = 30·4797 cm.	do. = 0·032809 ft.
1 yard = 91·4392 cm.	1 metre = 3·28087 ft.
1 mile = 1·60933 km.	1 kilometre = 0·62138 ml.

17. *Measurement of Area and Volume.*—We may notice here, though it is not necessary for our present purpose, the measurement of area and volume.

Any arbitrary area may be chosen as unit of surface or area. But the most convenient unit is the area of a square whose side is of unit length. The English units are accordingly the square yard, square foot, square inch, etc.; the French units, the square metre, square centimetre, etc.

1 sq. inch = 6·4516 sq. cm.	1 sq. centimetre = 0·1550 sq. in.
1 sq. foot = 929·01 sq. cm.	do. = 0·001076 sq. ft.
1 sq. yard = ·836113 sq. m.	1 sq. metre = 1·196 sq. yd.
1 sq. mile = 2·59 sq. km.	1 sq. kilometre = 0·3861 sq. ml.

Similarly, the most convenient unit of volume is that of a cube whose edge is of unit length. The English units are thus the cubic yard, cubic foot, etc.; the French

units are the cubic metre, cubic decimetre (called the litre), etc.

1 cu. inch = 16·387 cu. cm.	1 cu. cm. = 0·06102 cu. in.
1 cu. foot = 28316· cu. cm.	do. = $3·532 \times 10^{-5}$ cu. ft.
1 cu. yard = 0·764535 cu. m.	1 cu. metre = 1·308 cu. yd.

18. *Derived Units.*—A unit of a quantity of one kind which is thus defined by reference to the unit of a quantity of another kind is called a derived unit. The magnitude of such a unit will depend upon that of the simple, or arbitrarily chosen, unit, by reference to which it is derived. Thus it is clear that if our unit of length be increased two, three, four, etc., times, our unit of area will be increased four, nine, sixteen, etc., times respectively; or, generally, that the magnitude of the unit of area is directly proportional to the square of the magnitude of the unit of length. In symbols, if $[S]$ represent the magnitude of the unit of area, and $[L]$ that of the unit of length, $[S] \propto [L]^2$.

A statement of the mode in which the magnitude of a derived unit varies with the magnitudes of the simple units involved in it, is called a statement of the *dimensions* of the unit. The unit of area has thus the dimensions $[L]^2$.

19. Though this result is sufficiently obvious, we may obtain it by a method which we shall find useful when dealing with more complicated units. Let s be the area of a square whose side is l. Then $s = l^2$. Now (15) $s \propto 1/[S]$ and $l \propto 1/[L]$. Hence $[S] \propto [L]^2$.

20. The reader will find no difficulty in showing in a similar way, that the unit of volume has the dimensions $[L]^3$.

21. *Measurement of Angle.*—There are two *units of plane angle* in ordinary use, the degree and the radian.

The *degree* is the ninetieth part of a right angle; and its subdivisions are the minute, which is one sixtieth part of a degree, and the second, which is one sixtieth part of a minute. The *radian* is the angle subtended at the centre of a circle by an arc equal in length to the radius. As the circumference of a circle is 2π times the radius, the radian is equal to $360° \div 2\pi$, *i.e.*, to $57°{\cdot}29578\ldots$ or to $57°\ 17'\ 44''{\cdot}8$ nearly. It is subdivided decimally. The numerical measure of an angle in radians is often called its "circular measure." It is obvious that the angle subtended at the centre of a circle of radius r, by an arc of length a, is equal to a/r radians, and that consequently the magnitude of the radian is independent of the magnitude of the unit of length.

22. The *unit of solid angle* is the solid angle subtended at the centre of a sphere by a portion of its surface whose area is equal to the square of its radius. It may be called the *solid radian*. It follows that the solid angle subtended at the centre of a sphere of radius r, by a portion of its surface whose area is A, is A/r^2 solid radians, and that the magnitude of the solid radian is thus also independent of that of the unit of length.

23. *Motion.*—The motion of a point is its change of position in space. It is therefore completely described by a statement of the changes in the co-ordinates of the point. Motion is thus, like position, a relative conception.

24. *Rest.*—A point which is undergoing no change of position, whose co-ordinates therefore are not varying, is said to be at rest relative to the origin of co-ordinates or point of reference. In any case in which we speak of a body as being simply "at rest," it is assumed that the point of reference is known.

A "fixed point" or a "point fixed in space" is one which, during the time under consideration, is at rest

relatively to the point which has been chosen as point of reference. A line fixed in space is one containing fixed points.

25. *Relation of Motion to Time.*—The motion of a body is found to occupy time; and one important object of Kinematics is to compare the contemporaneous motions of different bodies, and to determine the laws according to which the changes in the co-ordinates of some bodies are related to the contemporaneous changes in the co-ordinates of others. As it is not possible for one observer to make many observations of the positions of bodies at the same instant, it is necessary, for the attainment of this object, to be able to describe instants of time, in order that the observations of different observers may be comparable.

26. *Description of Instants of Time.*—To describe the times of occurrence of events, it is only necessary that we should fix upon some series of periodically recurring events and keep a record of them. We may choose, for example, the daily passage across the meridian, of a known point in the heavens, say a "fixed" star. In that case, the time of the occurrence of an event would be described as between the n^{th} and the $(n+1)^{\text{th}}$ transits of this star. To make the description more definite, we may use a rapidly oscillating pendulum, and describe the event as occurring between the m^{th} and $(m+1)^{\text{th}}$ oscillations of the pendulum after the n^{th} transit of the fixed star. By thus selecting series of events occurring with sufficient frequency, it is possible to give our descriptions of instants of time as great precision as may be desirable.

27. "*Measurement*" *of Time.*—As we are thus able to describe instants, it is possible to record the magnitudes which quantities (*e.g.*, distances, etc.) have at definite instants, and therefore to compare the changes which

the positions of bodies may have undergone in any required interval of time.

28. In order to compare the contemporaneous motions of any number of bodies among one another, it is only necessary to compare the motion of each body with that of some one selected as a standard of comparison. In selecting a standard, it will save a great deal of labour if we choose a body whose motion is such that as many as possible of the laws of the motions of other bodies, when expressed in terms of its motion, are (1) simple, and (2) permanent, *i.e.*, independent of the date of their determination. To fix upon such a moving body, it is necessary to make observations of the positions of many bodies at short intervals during long periods of time, and to keep records of them. This has been done by astronomers, whose records extend over 2,500 years. Their observations show, that if the motions of other bodies are compared with the rotation (194) of the earth relative to the "fixed" stars, the laws of their motions take forms which are simpler and more permanent than if any other motion is taken as the standard. Hence by common consent the motion of the earth about its axis is taken as a standard with which other motions are compared.

29. When the law of the change of the position of a body, with reference to the rotation of the earth about its axis, is determined, we are said to have determined the law of its change of position with reference to time, successive rotations of the earth being assumed to occur in equal intervals of time. Whether they do so or not, we have no means of knowing, as we have no means of measuring time. But this form of speech, which assumes the possibility of measuring time, is conveniently short, and provided we keep in mind its real meaning, can lead to no error.

The period of the earth's rotation with reference to

the fixed stars, *i.e.*, the period between successive instants at which a fixed star is on our meridian, is called a *sidereal day*. When we employ the earth's rotation relative to the fixed stars as a standard motion, we may be said to employ the sidereal day as a unit of time.

30. Recent discussion of astronomical observations* seems to show that the laws of the motions of heavenly bodies would take simpler forms, and would be more permanent, if the standard motion were that of an ideal earth, rotating so that its rate of rotation would slowly gain on the rate of rotation of the actual earth. At what rate the ideal earth's rate of rotation should gain on that of the actual earth in order that these laws may take their simplest and most permanent forms, is not known. But the astronomical data are sufficiently definite to show that it is exceedingly small. This result is expressed in the language of time by saying that the sidereal day is increasing at a very slow rate.

31. It is found practically inconvenient to compare the motions of bodies directly with the rotation of the earth relative to the fixed stars. They are usually compared directly with the rotation of the earth relative to the sun; and the law, according to which the earth rotates relatively to the sun, having been determined in terms of its rotation relative to the fixed stars, they can thus be indirectly compared with the standard motion. In the language of time, it is found more convenient to measure time in terms of the solar day than of the sidereal day.—The solar day being a variable period, the mean solar day is chosen as practical unit. It is found to be equal to 1·002738... sidereal days.

32. It is frequently convenient to compare motions with some periodic motion of much greater frequency than

*See Thomson and Tait's "Treatise on Natural Philosophy," pt. II., § 830.

the rotation of the earth. In such cases the oscillation of a pendulum is chosen; for it is found that if a pendulum is kept in a constant physical condition, it will oscillate the same number of times in different sidereal days, and that in $1/n^{th}$ of a day (*i e.*, while the earth is making $1/n^{th}$ of a rotation) it will make $1/n^{th}$ of the number of oscillations made in a whole day. The *second* is the time of oscillation of a pendulum which oscillates 86,400 (*i.e.*, $24 \times 60 \times 60$) times in a mean solar day. The sidereal day contains 86,164 mean solar seconds. A *clock* is an instrument for maintaining a pendulum in oscillation and for counting its oscillations.

33. *Complexity of Motion.*—The motions of bodies may be of various degrees of complexity. The simplest form is that in which all points of the body move through equal distances in the same direction. Such a motion is called a *translation.* If, though the various points of the body maintain the same relative positions during the motion, they do not move through equal distances in the same direction, the motion is partly or wholly a *rotation.* If, finally, the points of the body do not maintain the same distances from one another during the motion, the motion consists partly of a *strain* or change of volume or form.

We shall see farther on that the action of a force upon a body usually affects the motion of the body in all these ways. It is convenient, however, to study the different kinds of motion separately, assuming bodies to have that kind of motion alone, which, for the time, we may wish to investigate.

CHAPTER II.

TRANSLATION:—IN GIVEN PATHS.

34. We have defined translation to be the motion which a body has when all its points move through equal distances in the same direction. If then the motion of one point is known, the translation of the body is known. Hence the study of the translation of a body is the same as the study of *the motion of a point.*

35. *Degrees of Freedom.*—The position of a point, as we have seen, is determined by three numbers, which may be measures of distance or of distance and angle. The motion is determined if the changes in these measures are known. Hence a point is said to have three degrees of freedom to move.

If the point be constrained to remain on a given surface its position can then be determined by two numbers, and it has therefore two degrees of freedom. One degree of constraint is said to have been introduced. The condition of constraint in this case is that the distance of the point from the surface shall be zero. If the point be constrained to remain on each of two surfaces it must remain on their line of intersection. Hence its position and its motion may be determined by one number, the distance or the change of distance from a given point in the line. It has one degree of freedom. Two degrees of

constraint have been introduced. A third degree of constraint, the condition for instance that the point remain on a third surface, will confine it to the point in which the three surfaces intersect: it has then no freedom.

Constraint is of course not necessarily applied in the way mentioned above. Thus the condition that a point shall maintain a given distance from a given fixed point restricts its motion to the surface of a sphere. A second degree of freedom is destroyed by constraining the point to move in a vertical plane, and it can now move only in the curve of intersection of the vertical plane and the sphere. If now it be so constrained that the line joining it with the fixed point maintains a constant inclination to a fixed line in the given vertical plane, the point has three degrees of constraint and consequently a definite position.

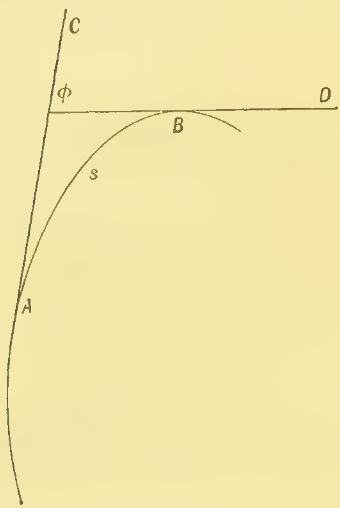
36. *Paths.*—The path of a moving point is the locus of its successive positions. It must be a continuous line, but may have any form whatever. We shall see farther on (295), however, that the path of a material particle (310) can undergo no abrupt changes of direction, unless indeed its motion cease and recommence; and we shall restrict ourselves to the study of paths which are possible for material particles.

The direction of such a path at any point is that of the tangent at that point.

37. *Curvature.*—The change of direction between any two points of a path lying wholly in one plane is called the *integral curvature* between these points. It is evidently measured by the angle between the tangents at these points.

The *mean curvature* between two points is the integral curvature between them divided by the length

of path intercepted by them. Thus, if AB is a portion of the path of a moving point, AC and BD being tangents at A and B respectively, inclined at the angle ϕ, then ϕ is the integral curvature between A and B; and, s being the length of the arc AB, ϕ/s is the mean curvature between A and B.



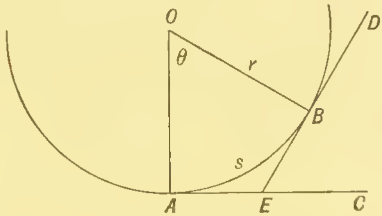
The mean curvature between A and B will in general have different values as B is taken nearer or farther from A. If it is the same whatever the position of A and B on the path, the curvature of the path is said to be uniform.

The mean curvature between A and B will have a finite value, however small the distance between them, in the case of the path of a material body. For, as there can be no abrupt changes of direction in the path, ϕ and s vanish together, and to an indefinitely small value of s corresponds an indefinitely small value of ϕ.

The limiting value of the mean curvature between A and B, when B is indefinitely near A, is called *the curvature at A*. The curvature at any point of a curve of uniform curvature is evidently equal to the mean curvature between any two points.

38. *The Curvature of a Circle.*—Let A, B be any two points on a circle whose centre is O, and AC, ED tangents at A, B respectively. The angles OBE and OAE are right angles, and hence the angle BEC is equal to the angle O. Hence, if θ stand for the angle O, r for the radius, and s for the length of the arc AB, the mean

curvature between A and B has the value θ/s, which, if θ is measured in radians, is (21) equal to $s/r \times 1/s$ or to



$1/r$. Since A, B are any two points on the circle, its curvature is uniform; and consequently both the mean curvature between any two points and the curvature at any point are measured by the reciprocal of the radius.

39. *The Curvature of any Plane Curve.*—As the curvature of a circle depends upon its radius only, a circle can always be found whose curvature is the same as the curvature of any given curve at any given point. That circle whose curvature is equal to the curvature of a given curve at a given point is called the *circle of curvature* of the curve at that point. Its radius is called the *radius of curvature*, and its diameter the *diameter of curvature*. If ρ is the radius of curvature, the curvature of the given curve at the given point is $1/\rho$. If a circle whose curvature is equal to that of the curve at the given point, be drawn touching the curve at the given point, the concavities of the two curves having the same aspect, its centre is called the *centre of curvature* of the curve at the given point; and any chord of the circle through the point of contact is called a *chord of curvature.*

40. The only plane curve whose curvature is uniform is clearly the circle. For every element or indefinitely small portion of such a curve coincides with an element of a circle of constant radius.

41. *Tortuosity.*—In the case of paths (called tortuous paths) not lying wholly in one plane, the nature of their

curvature may most readily be seen by imagining them to be polygons with an indefinitely large number of indefinitely short sides. Then any two adjacent sides must be in the same plane; but the planes containing pairs of adjacent sides are different at different parts of

the curve. Let AB, BC, CD be three sides of such a polygon. Then AB and BC are in one plane, and BC and CD are in one plane also; but the plane containing AB and BC is not the same as that containing BC and CD. The *osculating* plane of such a path at any point is the plane in which the portion of the path indefinitely near that point lies. In other words, it is the plane in which adjacent sides of the imaginary polygon lie. The osculating plane at B contains BA and BC, that at C contains CB and CD. Hence it passes from the position in which it contains AB and BC to that in which it contains BC and CD by rotating about BC. Therefore the osculating plane passes from its position at any point of a curve to its position at a neighbouring point by rotating about the tangent to the curve. The amount of this rotation (*i.e.*, the total angle through which the osculating plane rotates) between any two points of a curve is the *integral tortuosity* between them. The integral tortuosity divided by the distance of the points measured along the curve is the *mean tortuosity* between them. And the *tortuosity* at one of these points is the limiting value of the mean tortuosity when the second point is moved up towards the first. The consideration of tortuous paths is beyond the scope of this book.

42. *Speed.*—The *mean speed* of a moving point, during a given time, is the quotient of the length of its path traversed in the time, by the time: or, in other words, the mean rate of motion in the path during the time. If

s and s' are the initial and final distances (measured along the path) of the moving point from a fixed point in the path, t being the interval of time, the mean speed is thus $(s'-s)/t$. As the distance from the fixed point may be either increasing or diminishing, the mean speed is a quantity which has not magnitude merely, but also sign. Such a quantity is called a *scalar* quantity.

In general, the mean speed has different values for different intervals of time, and is said to be variable. In special cases in which it has the same value, whatever the interval of time to which it applies, the point is said to move with uniform speed. A point having such a motion obviously traverses equal portions of its path in equal times.

43. The *instantaneous speed* of a moving point (usually spoken of as the speed simply) at a given instant is the limiting value of the mean speed between that instant and another, when the interval of time between them is made indefinitely small. We shall see later on (295) that in an indefinitely short time a particle can traverse only an indefinitely small portion of its path. Hence the instantaneous speed of a particle has always a finite value.

It is clear that the instantaneous speed, at any instant, of a point whose speed is uniform, is equal to its mean speed during any period of time.

The speed of a moving point is usually called its velocity. To assist the beginner in keeping his ideas clear, it is better to restrict the term velocity to a more complex conception (92) to which it is also usually applied.

44. A quantity which varies with time is called a *fluent*, and the rate of its variation is called its *fluxion* or its *flux*. Thus the distance s measured along the

path of a moving point from a fixed point in the path, is a fluent; its speed is the fluxion or flux. When the phrase "rate of change" is used without further specification, the rate of change with time, or the fluxion, of the quantity under consideration is always meant. Newton denoted the fluxion by the symbol for the fluent with a small dot above it. Thus his symbol for the speed is $\dot{s}$.

45. *Measurement of Speed.*—We might choose any concrete speed, as for instance that of a ray of light *in vacuo*, as a unit. But if we keep to the units of length and time selected above, our unit of speed is determined by the definitions already given (42 and 43). For both the mean and the instantaneous speeds of a moving point were defined to be quotients of the value of a certain length by that of a certain time. If, then, v is the numerical value of the speed, s that of the length, and t that of the time, we have $v = s/t$. If now s and t are both unity, v must be unity also. Hence we have taken as unit of speed that of a point moving at the rate of one unit of length in one unit of time. Expressed in English units, it may be one foot per second, one mile per hour, etc.; in French units, one centimetre per second, one kilometre per hour, etc. The unit of speed is thus a derived unit, like the units of surface and volume.

46. *Systems of Units.*—The simple units (of length and time so far as we have gone), together with the units derived from them, constitute a system of units. Thus we have the foot-second system, consisting, so far as we have gone, of the foot, the second, the square foot, the cubic foot, the foot-per-second. We may have also the mile-hour system, the centimetre-second system, and as many others as there are sets of simple units. For scientific purposes the centimetre-second system is the most useful.

47. *Dimensions of the Unit of Speed.*—If $[V]$, $[L]$, $[T]$ denote the magnitudes of the units of speed, length, and time respectively, in terms of which v, s, and t of the formula $v=s/t$ (45) are expressed, we have (15)

$$v \propto 1/[V]; \;\; s \propto 1/[L]; \;\; t \propto 1/[T].$$

Hence $[V] \propto [L]/[T]$; *i.e.*, the magnitude of the derived unit of speed is directly proportional to the magnitude of the unit of length, and inversely proportional to the magnitude of the unit of time, involved in it. The dimensions of the unit of speed are thus $[L][T]^{-1}$.

48. If $[V]$, $[V']$ are the magnitudes of different similarly derived units of speed, and $[L]$, $[T]$, and $[L']$, $[T']$ those of the simple units involved in them respectively, we have

$$[V]:[V'] = [L]/[T]:[L']/[T'],$$

or

$$[V]/[V'] = [L]/[L'] \div [T]/[T'].$$

If therefore the magnitude of any derived unit of speed be expressed in terms of some other similarly derived unit of speed, and if the magnitudes of the simple units involved in the first be expressed in terms of those of the simple units involved in the second, the magnitude of the unit of speed thus expressed will be equal to the ratio of the magnitudes, also thus expressed, of the units of length and time.

49. If v and v' be numerical values of the same speed in terms of units whose magnitudes are $[V]$ and $[V']$, then (15)

$$v:v' = [V']:[V].$$

Hence, with the symbols of 48,

$$v:v' = [L']/[T']:[L]/[T].$$

If therefore the numerical value of a speed be given in terms of one set of units, its value can be determined in terms of any other set.

50. *Examples.*

(1) A point moving in a circle of 40 ft. radius makes 4·5 revolutions in 20 seconds. Show that the mean speed is 56·5...ft. per sec.

(2) A railway train runs from A to D, stopping at B and C. The distances are : A to B, 20 miles ; B to C, 5 miles ; C to D, 10 miles. It goes from A to B in 30 min., from B to C in 10 min., and from C to D in 14 min. It remains 2 min. at B and 10 min. at C. Find the mean speed (a) during the whole time, (b) between the times of leaving A and C, and (c) between the time of leaving B and that of arriving at D.

Ans. (a) 0·53..., (b) 0·48..., (c) 0·44... mile per min.

(3) The distance (s feet, measured along the path) of a moving point from a given point in its path, at any time (t seconds after the instant chosen as zero) being given by the formula $s=4+5t$, show that the mean speed for any interval and the instantaneous speed at any instant are both 5 ft. per sec. [To determine the instantaneous speed find the value of $(s'-s)/(t'-t)$ where t and t' and therefore s and s' differ by indefinitely small quantities.]

(4) The distance s of Example 3 being represented by the formula $s=5t+6t^2$, show that the mean speed between the beginning of the 10th and the end of the 12th second is 131 ft. per sec., and that the instantaneous speed at the end of the 10th second is 125 ft. per sec.

[To find the instantaneous speed at the end of t seconds, we have

$$s'-s=5(t'-t)+6(t'^2-t^2).$$

Hence $$(s'-s)/(t'-t)=5+6(t'+t)=5+12t,$$

since t and t' are indefinitely nearly equal.]

(5) Compare the magnitudes of the foot-second and the mile-hour units of speed.

The magnitudes of the units of length and time involved in these units of speed are : $[L]=1$ ft., $[T]=1$ sec., $[L']=1$ mile $=5280$ ft., $[T']=1$ hour $=3600$ sec. Hence (47) the magnitude of the ft.-sec. unit being $[V]$ and that of the mile-hour unit being $[V']$ we have

$$[V]:[V']=[L]/[T]\;:[L']/[T']=1:\frac{5280}{3600}.$$

Hence 1 ft. per sec. $=\frac{3600}{5280}$ mile per hour.

Otherwise, without using formulae, thus :

$$1 \text{ ft. per sec.} = \frac{1}{5280} \text{ mile per sec.}$$
$$= \frac{1}{5280} \text{ mile per } \frac{1}{3600} \text{ hour}$$
$$= \frac{3600}{5280} \text{ mile per hour.}$$

(6) How many cm.-sec. units of speed are equivalent to 20 ft.-sec. units ?

Ans. 609·594.

(7) Compare the centimetre per second with the mile per hour.

Ans. 1 mile per hour = 44·704 cm. per sec.

(8) Show that 1 kilometre per hour is equivalent to $27\cdot\dot{7}$ cm. per. sec.

(9) A speed of 20 ft. per sec. being a derived unit, and 14 inches being the unit of length involved in it, find the unit of time.

Here (48), $[V]$ = 20 ft. per sec. and $[L] = 14$ in. $= \frac{14}{12}$ ft. Hence, the magnitudes of these units being both expressed in terms of the units of the foot-second system,

$$[V] = [L]/[T],$$

and
$$[T] = [L]/[V] = \frac{14}{12 \times 20} \text{ sec.}$$

Otherwise, without using formulae, thus :

$$\text{Unit of speed} = 20 \text{ ft. per sec.}$$
$$= 20 \times 12 \text{ in. per sec.}$$
$$= \frac{20 \times 12}{14} \text{ units of length per sec.}$$
$$= 1 \text{ unit of length per } \frac{14}{20 \times 12} \text{ sec.}$$

Hence
$$\text{Unit of time} = \frac{14}{20 \times 12} \text{ sec.}$$

(10) One cm. per sec. being the unit of speed of a derived system and 1 min. the unit of time, show that 60 cm. is the unit of length.

(11) Reduce 25 ft. per min. to cm. per sec.

The magnitudes of the units involved (49) are : $[L]=1$ ft. $=30{\cdot}4797$ cm., $[T]=1$ min. $=60$ sec., $[L']=1$ cm., $[T']=1$ sec. respectively. Hence, if v' is the numerical value in cm. per sec.,

$$25 : v' = 1 : \frac{30{\cdot}4797}{60}.$$

Hence $$v' = \frac{25 \times 30{\cdot}4797}{60} \text{ cm. per sec.}$$

Otherwise, without using formulae, thus :

$$\begin{aligned} 25 \text{ ft. per min.} &= 25 \times 30{\cdot}4797 \text{ cm. per min.} \\ &= 25 \times 30{\cdot}4797 \text{ cm. per 60 sec.} \\ &= \frac{25 \times 30{\cdot}4797}{60} \text{ cm. per sec.} \end{aligned}$$

(12) Reduce 24 ft. per sec. to yds. per min.

Ans. 480 yds. per min.

(13) In 40 cm. per sec., how many miles per hour ?

Ans. 0·8947...

(14) Find the value in kilometres per hour of 10 yds. per sec.

Ans. 32·918...

(15) Compare the speeds, 14 miles per hour and 14 yds. per min.

Ans. The former is $29{\cdot}\dot{3}$ times the latter.

(16) One point traverses 50 ft. in 6 min., another 50 cm. in 6 sec. Compare their mean speeds.

Ans. Their ratio is 0·50799...

51. *Change of Speed.*—The change of speed during a given interval of time is the difference between its final and its initial values.

52. *Rate of Change of Speed.*—The *mean rate of change of speed* of a moving point during any given time is the quotient of the change of speed by the time. In general, the mean rate varies with the length of the interval of time to which it applies, and is thus said to be *variable.* In cases in which it has the same value,

whatever the interval of time to which it applies, it is said to be *uniform*.

53. *The instantaneous rate* of change of speed (called usually the rate of change of speed simply), at a given instant, is the limiting value of the mean rate between that instant and another, when the interval of time between them is made indefinitely small. We shall see farther on (295) that the speed of a body cannot undergo any sudden change. Hence the instantaneous rate of change of speed of a body can never have an infinite value.

In general, the instantaneous rate of change of speed varies from instant to instant. In cases in which the mean rate of change is uniform, the instantaneous rate is clearly both the same at all instants and equal to the mean rate for any interval.

54. Rate of change of speed may be either rate of increase or rate of decrease, and may be thus either positive or negative. Hence it is a quantity having both magnitude and sign, *i.e.*, a scalar quantity (42).

Rate of change of speed is usually called acceleration. This term is also applied, however, to a more complex conception, to which we shall restrict it, that beginners may not be confused. It is desirable that a name should be invented for the phrase "rate of change of speed." Hayward has proposed the term *quickening*.

55. According to Newton's notation (44), a rate of change of speed, being the fluxion of a speed, should be written $\overset{:}{s}$. It is usually written $\ddot{s}$.

56. *Measurement of Rate of Change of Speed.*—The definitions of rates of change of speed given above determine at once the unit to be employed in their measure-

ment. Whether mean or instantaneous, uniform or variable, they are quotients of a certain speed by a certain time. If a be the value of the rate of change, v that of the speed, and t that of the time, we have $a = v/t$. If then v and t are both unity, a must be unity also. Hence we have taken as our unit of rate of change of speed that of a point whose speed is changing at the rate of unit of speed per unit of time. The English unit of the foot-second system is thus 1 ft.-per-sec. per sec.; that of the mile-hour system, 1 ml.-per-hour per hour. Similarly, the French unit of the cm.-sec. system is 1 cm.-per-sec. per sec. The second "per second" is often omitted; but this shortened mode of specifying the unit is apt to be misleading.

57. *Dimensions of Rate of Change of Speed.*—We have seen (56) that $a = v/t$. If now $[A]$ denote the magnitude of the unit of rate of change of speed, $[V]$ and $[T]$ those of the units of speed and of time respectively, we have (15)

$$a \propto 1/[A]\,;\ v \propto 1/[V]\,;\ t \propto 1/[T].$$

Hence $$[A] \propto [V]/[T].$$

But (47) $$[V] \propto [L]/[T].$$

Hence $$[A] \propto [L]/[T]^2,$$

or $$[A] \propto [L][T]^{-2},$$

i.e., the magnitude of the unit of rate of change of speed is directly proportional to the magnitude of the unit of length, and inversely proportional to the square of the magnitude of the unit of time.

58. As in the case of speed (48), so also in that of rate of change of speed, it may be shown that, if $[A]$, $[L]$, and $[T]$ are the magnitudes of the units of one derived system, and $[A']$, $[L']$, $[T']$ those of another similarly derived system,

$$[A]/[A'] = [L]/[L'] \div [T]^2/[T']^2;$$

i.e., if the magnitudes of the units of rate of change of speed, of length, and of time, of one system of derived units be expressed in terms of the magnitudes of the corresponding units of another similarly derived system, the magnitude of the unit of rate of change of speed will be equal to the magnitude of the unit of length divided by the square of the magnitude of the unit of time.

59. *Examples.*

(1) A point has at a given instant a speed of 4 ft. per sec.; and, after 8 sec., one of 20 ft. per sec. Find (a) the integral change of speed, and (b) the mean rate of change.

Ans. (a) 16 ft. per sec.; (b) 2 ft.-per-sec. per sec.

(2) If a point which moves in a curve traverse in t units of time after zero of time, an arc whose length $s = 2t + 3t^2 + 4t^3$, find (a) the instantaneous speed, and (b) the instantaneous rate of change of speed, at the end of the 5th second.

Ans. (a) 332 units of length per unit of time; (b) 126 units of speed per unit of time.

(3) If the formula of Ex. 2 had been $s = a/t + bt^2$ (a and b being constants), show that the instantaneous speed and rate of change of speed at the end of t units of time would have been $2bt - a/t^2$ and $2(a/t^3 + b)$ respectively.

(4) If the formula of Ex. 2 had been $s = at + bt^2$, show that the rate of change of speed would have been uniform.

(5) Find the number expressing the uniform rate of change of speed of a train which, 5 minutes after starting, is moving at the rate of 40 mls. per hour.

Ans. 480 mls.-per-hour per hour.

(6) Find how many kilometre-hour units of rate of change of speed are equivalent to 392 ft.-min. units.

The magnitudes of the units involved in these systems are: $[L] = 1$ kilometre; $[T] = 1$ hour; $[L'] = 1$ ft. $= 0{\cdot}00030\ldots$ kilom., $[T'] = 1$ min. $= \frac{1}{60}$ hour. Hence, if a be the equivalent required, we have,

by 57, employing a formula similar to that of 49, viz.,

$$a : a' = [L']/[T']^2 : [L]/[T]^2,$$

$$a : 392 = 0{\cdot}0003 \times 60^2 : 1.$$

Hence $\qquad a = 392 \times 60^2 \times 0{\cdot}0003$ kilom.-per-hour per hour.

Otherwise, without using formulae, thus: 392 ft.-min. units of rate of change of speed

$= 392$ ft.-per-min. per min.

$= 392 \times 0{\cdot}0003$ kilom.-per-min. per min.

$= 392 \times 0{\cdot}0003 \times 60$ kilom.-per-hour per min.

$= 392 \times 0{\cdot}0003 \times 60^2$ kilom.-per-hour per hour.

(7) Compare the foot sec. and the yd.-min. units of rate of change of speed.

Ans. 1 ft.-sec. unit = 1,200 yd.-min. units.

(8) How many cm.-sec. units of rate of change of speed in 1 mile-min. unit?

Ans. 44·704....

(9) The rate of change of speed of a falling body (32·2 ft.-sec. units) and 1 pole ($5\frac{1}{2}$ yds.) being the units of rate of change of speed and of length respectively of a derived system, find the unit of time. [See 58 and 50 (9).]

Ans. 0·7... sec.

(10) The unit of speed of a derived system being the speed of a point in the earth's equator (the earth being supposed a sphere of 4,000 mls. radius), and the unit of time the month (30 days), compare the unit of rate of change of speed with the ft.-sec. unit.

Ans. It is equal to 0·00059... ft.-sec. unit.

(11) Reduce 101 metre-min. units of rate of change of speed to cm.-sec. units.

Ans. 2·8... cm.-sec. units.

(12) Express (*a*) in cm.-sec. units, and (*b*) in kilom.-hour units, a rate of change of speed of 90 ft.-per-sec. per sec.

Ans. (*a*) 2,743·17...; (*b*) 355,515·6....

60. *Motion under Given Rates of Change of Speed.*—We have seen that the rate of change of speed of a point

or a particle may be uniform or variable. Its value may be the same at all instants of any interval of time, and therefore at all points of the path occupied during that time, or it may vary from point to point of the path and therefore from instant to instant. And, when the rate of change is variable, it may vary according to different laws. For example, it may vary directly as the distance of the moving point from a fixed point in its path (the distance being measured along the path), or inversely as this distance, or directly (or inversely) as the square of this distance, and so on.

61. *Case I.—The Rate of Change of Speed being Zero.*—In this case the speed is uniform. Hence the instantaneous speed at any instant is equal to the mean speed during any interval (43), and therefore to the quotient of the length of path traversed in the time, by the time. If therefore v is the speed, and s the length of path traversed in t units of time, we have (42) $v=s/t$, and therefore $s=vt$.

62. *Examples.*

(1) A point has a uniform speed of 10 ft. per sec. Find the length of path traversed in 1 hour.

Ans. $6\cdot\dot{8}\dot{1}$ mls.

(2) A point moves for 1 min. with uniform speed in a circle of 30 decimetres radius, traversing in that time an arc of 0·4 radian. Find the speed.

Ans. 2 cm. per sec.

(3) Find the time required by a point moving with a uniform speed of 40 cm. per sec. to traverse a path 20 metres long.

Ans. 50 sec.

63. *Case II.—The Rate of Change of Speed being Uniform.*—In this case the instantaneous rate at any instant is equal to the mean rate during any interval

(53). Let v_0, v be the initial and final values of the speed of the moving point during the time t, a being the rate of change of speed. Then $a=(v-v_0)/t$, and $v=v_0+at$. Hence the final speed is expressed in terms of the initial speed, the rate of change of speed, and the time.

64. As the speed increases uniformly with the time, its value at the middle of the interval is the arithmetic mean of its values at instants τ seconds before, and τ seconds after, the middle of the interval. Its value at the middle of the interval is therefore equal to the arithmetic mean of its initial and final values, and also to the mean speed during the interval. The mean speed is therefore equal to

$$\tfrac{1}{2}[v_0+(v_0+at)]=v_0+\tfrac{1}{2}at.$$

Hence, if s denote the distance measured along the path between the initial and final positions of the moving point,

$$s=(v_0+\tfrac{1}{2}at)t=v_0t+\tfrac{1}{2}at^2.$$

Hence this distance also is expressed in terms of the initial speed, the rate of change of speed, and the time.

65. Eliminating t between these expressions for v and s (63 and 64) we find

$$v^2=v_0{}^2+2as.$$

Hence the final speed is expressed in terms of the initial speed, the rate of change of speed, and the length of path between the initial and final positions.

66. By means of the above equations, if the initial position of a point moving in a given path, its initial speed, and its uniform rate of change of speed be given, its final position and its final speed after any interval of time can be determined.

67. *Examples.*

(1) A railway train is moving with a speed of 20 mls. per hour, and is increasing its speed uniformly at the rate of 10 mls.-per-hour per hour. Find (a) its speed after $1\frac{1}{2}$ hours, and (b) the distance traversed in that time. [For (a)—Data: $v_0 = 20$ mls. per hour; $a = 10$ mls.-per-hour per hour; $t = 1{\cdot}5$ hours. To be determined: v. And (63), $v = v_0 + at$. For (b)—Data: as above. To be determined: s. And (64), $s = v_0 t + \frac{1}{2}at^2$.]

Ans. (a) 35 mls. per hour; (b) $41\frac{1}{4}$ mls.

(2) A railway train, moving at 50 mls. per hour, has the brakes put on, and its speed diminishes uniformly for 1 minute, when it is found to have a speed of 20 mls. per hour. Find (a) its rate of change of speed, and (b) the distance traversed in the time.

Ans. (a) $-1{,}800$ ml.-hour units; (b) $\frac{7}{12}$ ml.

(3) A point whose speed is initially 20 m. per sec., and is diminishing at the uniform rate of 50 cm.-per-sec. per sec., moves in its path until its speed is 120 m. per min. Find the length of path between the initial and final positions. [Data: $v_0 = 20$ m. per sec.; $a = -50$ cm.-per-sec. per sec. $= -0{\cdot}5$ m.-per-sec. per sec.; $v = 120$ m. per min. $= 2$ m. per sec. And (65), $v^2 = v_0{}^2 + 2\,as$.]

Ans. 396 m.

(4) A point has a uniform rate of increase of speed of 20 cm.-per-sec. per sec. and an initial speed of 30 cm. per sec. Find (a) the speed after 16 sec.; (b) the time required to traverse 300 cm.; (c) the change of speed in traversing that distance.

Ans. (a) 350 cm. per sec.; (b) $(\sqrt{129} - 3)/2$ sec.; (c) $10(\sqrt{129} - 3)$ cm. per sec.

(5) If in Ex. (4) the speed be decreasing instead of increasing, find (a) the distance from the starting point to the turning point, (b) the distance from the starting point after 10 sec.; (c) the length of path traversed during the time in which the speed changes to 60 cm. per sec.; (d) the time required by the moving point to return to the starting point. [To find (a), note that the speed

at the turning point is zero; and to find (d), note that on the return to the starting point the distance therefrom is zero.]

Ans. (a) 22·5 cm.; (b) 700 cm.; (c) 112·5 cm.; (d) 3 sec.

(6) A particle moves round a closed curve with a uniform rate of change of speed. In the m^{th}, n^{th}, and p^{th} seconds, it describes a, b, and c circuits respectively. Show that, the initial speed being zero,

$$a(n-p)+b(p-m)+c(m-n)=0.$$

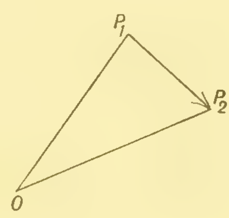
N.B.—The reader should solve the above problems also without using formulae. Others of a similar kind will be found in 141.

68. *Case III.—The Rate of Change of Speed Variable.*—Cases in which the rate of change of speed varies with the position of the moving point in its path can be discussed with greater advantage at a later stage (see 158, 164).

CHAPTER III.

TRANSLATION :—DISPLACEMENTS, VELOCITIES, ACCELERATIONS.

69. *Displacements.*—The change of position of a point, considered without reference to the intermediate positions occupied by it, is called its displacement. The displacement in any time is thus completely determined, if we have data by the aid of which the point may be brought from its initial to its final position. Let P_1 and P_2 be the initial and final positions of the point relative to O. The displacement is completely determined if the direction and length of the straight line P_1P_2 are known. A displacement is thus a quantity having both magnitude and direction.



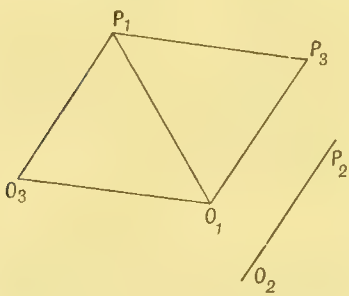
70. Any such quantity is called a *vector*. It may be completely represented by a straight line. For the length of the line may be made proportional to the magnitude of the vector, and its direction may be made the same as that of the vector. As a line, however, may represent either of two opposite directions, it is necessary to indicate, in the case of any given representative line, which of the two possible directions is that of the vector which it represents. For this purpose an arrow-head is frequently employed in diagrams, and in naming a line that letter is always placed first, which stands at the

end of the line from which the vector is directed. Thus a displacement P_1P_2 means one in the direction of the straight line P_1P_2 from P_1 towards P_2. When we speak of a vector as being represented by a line, complete representation as to both magnitude and direction is intended.

It should be noted that all lines which have the same length and direction represent the same vector. It is not necessary that a line intended to represent a vector should be drawn from any particular point.

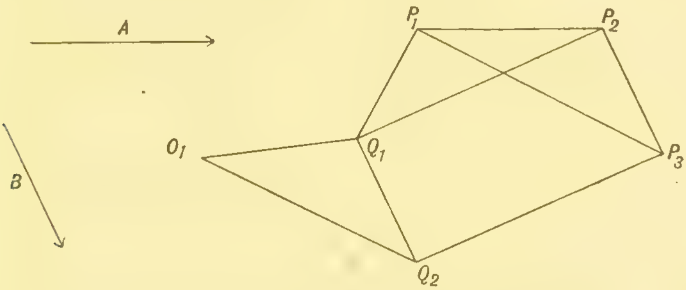
71. *Change of the Point of Reference.*—A displacement being a change of position can be described only by reference to some chosen point. As it is frequently necessary to change the point of reference, the following propositions will be found useful.

72. *Prop. I.*—A change in the relative positions of two points P and O may be regarded as either a displacement of P relative to O or an equal and opposite displacement of O relative to P.—Let P_1 and O_1 be the initial positions of two moving points P and O (the point of reference is not marked in the figure), and let P_2 and O_2 be their final positions. From O_1 draw O_1P_3 equal to and codirectional with O_2P_2. Then P_1P_3 represents the displacement of P relative to O. From P_1 draw P_1O_3 equal to and codirectional with P_2O_2. Then O_1O_3 represents the displacement of O relative to P. Now, since P_1O_3 is equal and parallel to P_3O_1, P_1P_3 is equal and parallel to O_1O_3, and they are drawn in opposite directions. Hence the above proposition.



73. *Prop. II.*—Given the displacement of a point P relative to a point Q, and that of Q relative to a point O: to find the displacement of P relative to O.—Let A and

B represent the given displacements of P and Q respectively, and let O_1, P_1, Q_1, be the initial positions of O, P,

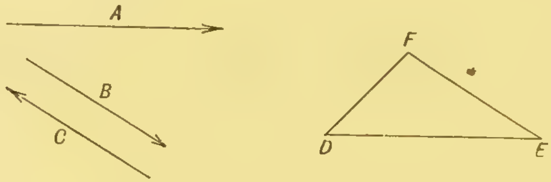


Q. Draw P_1P_2 and Q_1Q_2 equal to and codirectional with A and B respectively. Join Q_1P_2 and complete the parallelogram Q_1P_3.

P's final position relative to Q is given by the line Q_1P_2. Whatever, then, Q's final position relative to O may be, P's must be distant from it by the length Q_1P_2 and in the direction Q_1P_2. Now Q_2 is Q's final position relative to O. Hence P_3 is P's final position relative to O, for Q_2P_3 is equal to and codirectional with Q_1P_2. And P_1, P_3 being P's initial and final positions relative to O, P_1P_3 is its displacement relative to O. Now P_1P_2 and P_2P_3 represent P's displacement relative to Q and that of Q relative to O, respectively. Hence, if two sides of a triangle, taken the same way round, represent the displacements of P relative to Q and of Q relative to O respectively, the third side taken the opposite way round will represent the displacement of P relative to O.

It follows that if two sides of a triangle, taken the same way round, represent one the displacement of P relative to O, and the other that of O relative to P, the third side must represent the displacement of P relative to P, and must therefore be of length zero. Hence the lines representing the displacements of P relative to O and of O relative to P must be equal and opposite, as proved in Prop. I. (72).

74. *Prop. III.*—Given the displacements of P and Q relative to O; to find that of P relative to Q.—Let A and B represent the respective given displacements. Then (72), if C be drawn equal to B and in the opposite direction, A and C will be the displacements of P relative to O and



of O relative to Q, respectively. From any point D draw DE equal to and codirectional with A. From E draw EF equal to and codirectional with C. Then (73) DF will represent the displacement of P relative to Q.

Hence, if two sides of a triangle taken the same way round represent the displacement of P relative to O, and one equal and opposite to that of Q relative to O, respectively, the third side, taken the opposite way round, will represent the displacement of P relative to Q.

In the special case in which A and B have the same direction the point F is on the line DE, and it is obvious that the displacement of P relative to Q is equal to that of P relative to O *minus* that of Q relative to O.

75. *Examples.*

(1) Two railway trains run on parallel roads, the one 5 miles northwards, the other 6 miles southwards. Find the displacement of the latter relative to the former.

Ans. 11 miles southwards.

(2) Two trains run, the one north-eastwards a distance of 20 miles, the other south-eastwards through the same distance. Find the displacement of the former relative to the latter.

Ans. 28·28... miles in a northerly direction.

(3) A's displacement relative to B is 10 ft. westward. C's displacement relative to B is 20 ft. in a direction 30° west of south. Show that A's displacement relative to C is 17·32... ft. northward.

(4) The point A moves a distance of 3 ft. in a given direction, relatively to a point O. Another point B moves, relatively to O, 4 ft. in a direction at right angles to the direction of A's displacement. Find the displacement of A relative to B.

Ans. 5 ft. in a direction inclined $\sin^{-1}\frac{4}{5}$ to that of A's displacement.

(5) Two trains A and B start from the same point and run, A 10 miles northwards, and B 8 miles north-eastwards. Find B's displacement relative to A.

Ans. 7·1318 miles in a direction 37° 30′·96 S. of E.

(6) A's displacement relative to B, is 13 miles southwards, and relative to C, 4 miles westwards. C is initially 10 miles south of B. Find C's final position relative to B.

Ans. Distance 23·34... miles; direction 9° 51′·9 E. of S.

(7) Two points move in the circumferences of equal circles (radius = 2 ft.) which are in contact. Both start from the point of contact. The one moves through a semicircle, the other through a quadrant. Find the displacement of either relative to the other.

Ans. $2\sqrt{10}$ ft. in a direction inclined $\tan^{-1}3$ to the common tangent.

(8) Find the displacement of the end of the minute hand of a clock relative to the end of the hour hand (both minute and hour hands being 6 inches in length) between 3 and 3.30 o'clock.

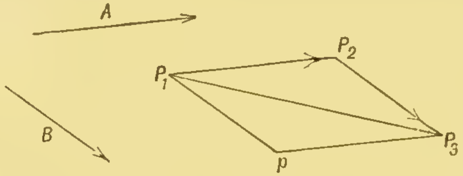
Ans. $6(6 - 2\cos 15^\circ - 4\sin 15^\circ)^{\frac{1}{2}}$ in.; direction inclined to the final direction of minute hand, $\sin^{-1}\{2\sin^2 7^\circ{\cdot}5/(6 - 2\cos 15^\circ - 4\sin 15^\circ)^{\frac{1}{2}}\}$.

(9) A wheel of 1 ft. radius rolls on a horizontal road turning through an angle of $\pi/2$ radians. Find the displacement of the point of the wheel initially in contact with the road relative to the point diametrically opposite to it.

Ans. $2\sqrt{2}$ ft.; direction inclined $\pi/4$ radians to the vertical.

76. *Composition of Successive Displacements.*—A point P undergoes given successive displacements, relative to the same point; it is required to determine the resulting (or *resultant*) displacement.

Case I.—Two Displacements.—Let A, B represent the displacements, and let P_1 be the initial position of the



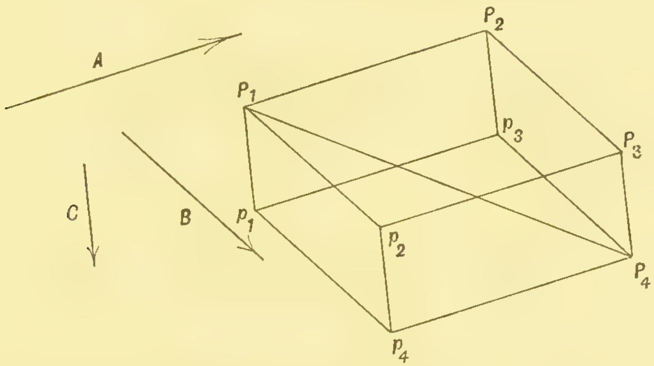
point. Draw P_1P_2 equal to and codirectional with A, and from P_2 draw P_2P_3 equal to and codirectional with B. Join P_1P_3. As the point is displaced first from P_1 to P_2, and then from P_2 to P_3, P_3 is its final position, and P_1P_3 is its resultant displacement. Complete the parallelogram pP_2. Then $P_1p = P_2P_3 = B$; and $pP_3 = P_1P_2 = A$. Hence the point P reaches the same final position whatever the order in which it undergoes the displacements A and B.

P_1P_3 is the third side of the triangles $P_1P_2P_3$, P_1pP_3, whose other sides P_1P_2 and P_2P_3 is the one triangle, and pP_3 and P_1p in the other, represent the displacements A and B respectively. Hence, if two sides of a triangle taken the same way round represent the two successive displacements of a moving point, the third side taken the opposite way round will represent the resultant displacement.

Also P_1P_3 is the diagonal of the parallelogram pP_2 through the point of intersection of the adjacent sides P_1P_2, P_1p, which represent the two successive displacements A and B. Hence, if two successive displacements of a point be represented by two adjacent sides of a parallelogram, taken opposite ways round, the diagonal of the parallelogram through their point of intersection will represent the resultant displacement.

Case II.—More than Two Displacements.—Let A, B, C represent the successive displacements, P_1 being the

initial position of the moving point. Draw P_1P_2, P_2P_3, P_3P_4 equal to and codirectional with A, B, and C respectively. Join P_1P_4. Then P_4 being the final position of



the moving point, P_1P_4 is the resultant displacement. The same construction is applicable to any number of displacements. If A, B, C, etc., are all in one plane, $P_1P_2P_3P_4\ldots$ is a plane polygon; if not, it is a *gauche* polygon.

It is clear that the same point P_4 is reached in whatever order the displacements occur. For, if the parallelogram P_2P_4 be completed, and then the parallelograms p_1P_2, p_1P_4, and p_1p_2, it follows from the equality and parallelism of the opposite sides of parallelograms that the line p_2P_3 will complete the parallelopiped P_2p_4, and that the six sets of displacements thus laid down, by which P_4 may be reached, are the displacements A, B, C taken in all possible orders. And a similar construction may be made whatever the number of displacements.

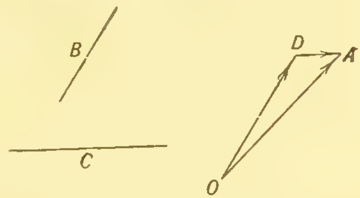
Hence, if any number of successive displacements of a moving point, in any directions whatever, be represented by $n-1$ of the sides of a polygon, taken the same way round, the resultant displacement will be represented by the n^{th} side taken the opposite way round.

77. It follows from the last proposition that a given displacement may be resolved into any number of successive displacements, provided these displacements can be represented by $n-1$ of the sides of a polygon, taken the same way round, by the n^{th} side of which, taken the opposite way round, the given displacement is represented. In the special case in which the successive displacements have directions parallel to the given displacement, it is clear that their algebraic sum must be equal to the given displacement.

78. *Composition of Simultaneous Displacements.*—A point undergoes simultaneously, given displacements relative to the same reference-point; it is required to determine the resultant displacement. Simultaneous displacements of a point are usually called *component displacements.*

Let A, B, C, etc., be the component displacements, and let each of them (77) be resolved into n successive indefinitely small displacements in its own direction. Let the magnitudes of these displacements be a_1, a_2, etc., b_1, b_2, etc., c_1, c_2, etc., respectively. Then (76) the same final position will be reached whether the point undergo successively the displacements A, B, C, etc., or undergo the successive displacements a_1, b_1, c_1, etc., a_2, b_2, c_2, etc., and so on. But a_1, a_2, etc., being indefinitely small, the successive occurrence of the displacements a_1, b_1, c_1, etc., a_2, b_2, c_2, etc., and so on, is the same as the simultaneous occurrence of the displacements A, B, C, etc. Hence the same final position is reached when A, B, C, etc., occur simultaneously as when they occur successively. Consequently the propositions established in 76 for successive displacements apply also to simultaneous or component displacements. These propositions when formulated for simultaneous displacements are usually called the *triangle,* the *parallelogram* and the *polygon, of displacements.*

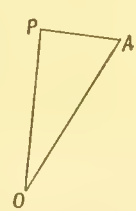
79. *Resolution of Displacemnets.*—A displacement and two straight lines being given, all in the same plane, to find two displacements parallel to these lines, of which the given displacement is the resultant.—Let OA be the given displacement, B and C the given lines. From O and A draw lines parallel to B and C respectively, meeting in D. Then, by 78, the displacement OA is the resultant of the component displacements OD and DA, and these displacements are parallel to the given lines.



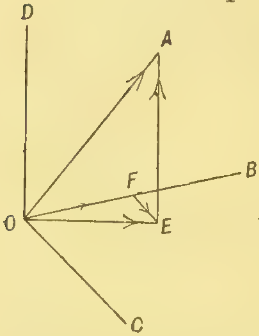
80. When the components, in given directions, of a given displacement are thus determined, the given displacement is said to be resolved into components in those directions.

Displacements are frequently resolved in directions which are at right angles to one another, in which case the components are called rectangular components. When we speak of *the* component of a displacement in a given direction, we mean its rectangular component in that direction. It is clear that the rectangular component of a displacement in any direction is the (orthogonal) projection of the displacement on any straight line in that direction.

81. To resolve a given displacement into two components, one being in a plane parallel to a given plane, and the other having no component (rectangular) in that plane. OA being the given displacement, draw from A a line AP perpendicular to the given plane, and from O a line OP perpendicular to AP and meeting it in P. OP is in a plane parallel to the given plane; and PA has no rectangular component in that plane. OP is clearly equal to the projection (orthogonal) of OA on the given plane.



82. The components of a given displacement, in three directions which are not all in the same plane, may also be found.—Let OA be the given displacement, and OB, OC, OD lines having the given directions. From A draw AE parallel to OD and meeting the plane of OB and OC in E. Join OE, and through E draw EF parallel to OC and meeting OB in F. Then (78) OE and EA are components of OA; and OF and FE are components of OE. Hence OF, FE, and EA are components of OA; and they are in the given directions.



The special case, in which each of the three directions OB, OC, OD is at right angles to the plane of the other two, is of great importance. In this case the components are adjacent edges of a rectangular parallelopiped of which OA is the diagonal through their intersection.

83. The resultant of two given displacements is equal to the algebraic sum of their components in its direction. Let OA, AB be the given displacements, and OB therefore the resultant displacement. From A draw AC perpendicular to OB. Then OC and CB are the components in the direction of OB, of OA and AB respectively. In Fig. 1, the components OC and CB have the same direction, and we have also $OB = OC + CB$. In Fig. 2, OC and CB have opposite directions, and we have also $OB = CB - CO$. Hence the displacement OB is equal to the algebraic sum of the components OC and CB, in the direction of OB, of the displacements OA and AB.

Fig. 1

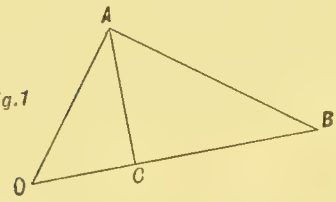
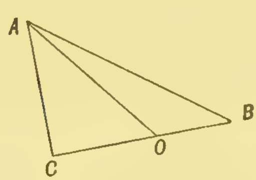
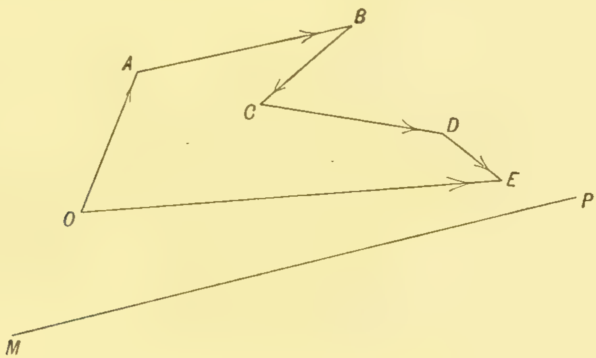


Fig. 2



84. The component in a given direction, of the resultant of any number of displacements in any directions whatever, is equal to the algebraic sum of their components in the same direction. Let OA, AB, BC, CD, DE repre-



sent the given displacements. Then (78) OE represents their resultant. Now the projection of OE on any line MP is equal (8) to the algebraic sum of the projections of OA, AB, BC, CD, DE. Hence (80) the above proposition is proved.

The proposition of 83 is clearly a special case of the above proposition.

85. *Trigonometrical Expression for the Resultant.*—The magnitude and inclination of component displacements being given, to find expressions for the magnitude and direction of the resultant.

First, when there are two given components.—Let A and B be the two components, their magnitudes being d_1 and d_2, and their inclination θ. θ may be an acute angle (Fig. 1) or an obtuse angle (Fig. 2). Let P_1 be the initial position of the point. Draw P_1P_2 and P_2P_3 equal to and codirectional with A and B respectively. Then (78) P_1P_3 is the resultant. Produce P_1P_2 to O. Then OP_2P_3 is the angle θ. From P_3 draw P_3Q perpendicular to P_1O.

To determine the magnitude of the resultant, we have from Geometry

(Fig. 1) $P_1P_3^2 = P_1P_2^2 + P_2P_3^2 + 2P_1P_2 \,.\, P_2Q$,

(Fig. 2) $P_1P_3^2 = P_1P_2^2 + P_2P_3^2 - 2P_1P_2 \,.\, P_2Q$.

Now in Fig. 1

$$P_2Q = P_2P_3 \cos\theta,$$

and in Fig. 2

$$P_2Q = P_2P_3 \cos(\pi - \theta) = -P_2P_3 \cos\theta.$$

Hence in both cases

$$P_1P_3^2 = P_1P_2^2 + P_2P_3^2 + 2P_1P_2 \,.\, P_2P_3 \cos\theta,$$

and, writing R for P_1P_3 and d_1, d_2 for the components,

$$R^2 = d_1^2 + d_2^2 + 2d_1d_2 \cos\theta.$$

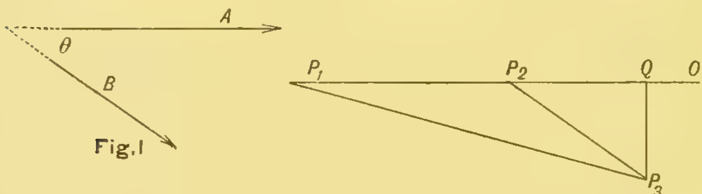


Fig. 1

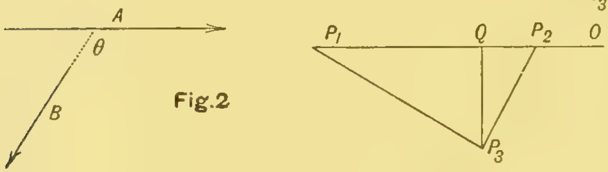


Fig. 2

To determine the direction of the resultant, we may find its inclination to one or other of the components, either the angle P_1 which we may call α, or the angle $P_1P_3P_2$ which we may call β. For this purpose we have, from Trigonometry,

$$\sin P_1 : \sin P_1P_2P_3 = P_2P_3 : P_1P_3.$$

Now in both the above cases the angle $P_1P_2P_3$ is equal to $(\pi - \theta)$. Hence

$$\sin\alpha : \sin\theta = d_2 : R,$$

and

$$\sin\alpha = \frac{d_2}{R} \sin\theta.$$

Similarly $$\sin\beta=\frac{d_1}{R}\sin\theta.$$

It follows that the displacement R has in the directions inclined a and β to its direction respectively, components whose magnitudes are

$$d_1=R\sin\beta/\sin(a+\beta),$$
$$d_2=R\sin a/\sin(a+\beta).$$

86. Of these general results, the following are important special cases.*

Case I.—The displacements equal. Let both be called d.

Then $$R^2=2d^2(1+\cos\theta)=4d^2\cos^2(\theta/2).$$

Hence $$R=2d\cos(\theta/2).$$

Also $$\sin a=\frac{d}{R}\sin\theta=\frac{d\sin\theta}{2d\cos(\theta/2)}=\sin\frac{\theta}{2}.$$

Hence $a=\theta/2$. Similarly $\beta=\theta/2$.

Case II.—The displacements equal and their inclination 120°. Then, by Case I.,

$$R=2d\cos 60^\circ=d,$$

and $$a=\beta=60^\circ.$$

Case III.—The displacements in the same direction, *i.e.*, $\theta=0$. Hence $\cos\theta=1$.

Therefore $$R^2=d_1{}^2+d_2{}^2+2d_1d_2,$$

and $$R=d_1+d_2.$$

Case IV.—The displacements in opposite directions, *i.e.*, $\theta=180^\circ$. Hence $\cos\theta=-1$.

Therefore $$R^2=d_1{}^2+d_2{}^2-2d_1d_2,$$

and $$R=d_1-d_2.$$

Displacements in opposite directions being considered of opposite sign, Cases III. and IV. may be generalized

* The reader should obtain the results of these special cases directly.

thus: The resultant of component displacements in the same straight line is their algebraic sum.

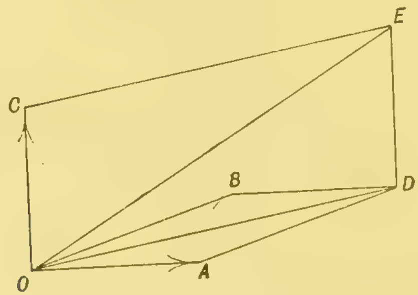
Case V.—The displacements at right angles to one another, *i.e.*, $\theta = 90°$. Hence $\cos\theta = 0$ and $\sin\theta = 1$.

Therefore $$R^2 = d_1^2 + d_2^2,$$
and $$R = (d_1^2 + d_2^2)^{\frac{1}{2}}.$$
Also $$\sin\alpha = d_2/R,$$
and $$\sin\beta = d_1/R.$$

In this case $\alpha + \beta = 90°$, and therefore $\sin\beta = \cos\alpha$. Hence $\tan\alpha = d_2/d_1$. Hence also the component (rectangular) of a displacement R in a direction inclined at the angle α to the direction of R is equal to $R\cos\alpha$.

87. *Secondly, when there are more than two given components.*—The magnitudes and directions of three or more component displacements being given, expressions may be found for the magnitude and direction of the resultant, by finding, first, the resultant of any two, then the resultant of this first resultant and a third, then the resultant of this second resultant and a fourth, and so on, until all the component displacements have been compounded.

88. An important special case of 87 is the composition of three displacements, the direction of each of which is



perpendicular to the plane of the other two. Let OA, OB, OC be the three displacements, the angles AOB,

AOC, and BOC being right angles. Complete the rectangle AB and draw the diagonal OD. Then OC, being at right angles to OA and OB, is also at right angles to OD. Complete the rectangle CD and join OE. Then OD is the resultant of OA and OB, and OE that of OD and OC. OE therefore is the resultant of all three. Now, by Geometry, $OE^2 = OA^2 + OB^2 + OC^2$. Hence, calling the resultant R, and the components d_1, d_2, d_3, we have

$$R = (d_1^2 + d_2^2 + d_3^2)^{\frac{1}{2}}.$$

The direction of OE is known, if either the angles AOD and DOE (ϕ and χ), or the angles AOE, BOE, COE (α, β, γ) are known. These angles may be expressed in terms of the magnitudes of the given displacements. For we have

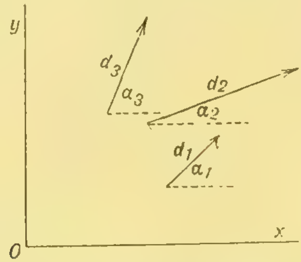
$$\begin{aligned}
\cos\phi &= d_1/OD = d_1/(d_1^2 + d_2^2)^{\frac{1}{2}},\\
\cos\chi &= OD/OE = (d_1^2 + d_2^2)^{\frac{1}{2}}/R,\\
\cos\alpha &= d_1/R,\\
\cos\beta &= d_2/R,\\
\cos\gamma &= d_3/R.
\end{aligned}$$

89. It follows that the components d_1, d_2, d_3, into which a given displacement R may be resolved, in three directions which are at right angles to one another and are inclined to the direction of R at the angles α, β, γ, are

$$d_1 = R\cos\alpha\,; \quad d_2 = R\cos\beta\,; \quad d_3 = R\cos\gamma.$$

90. *Analytical Expression for the Resultant* of any number of component displacements. Convenient expressions for the magnitude and direction of the resultant of any number of component displacements, may be obtained by resolving the given components in rectangular directions which are the same for all, adding the components in these directions, and finding the resultant by 86 (V.) or 88.

First, let the given displacements be all in one plane.—Let d_1, d_2, d_3, etc., be the given displacements. Take two lines Ox, Oy at right angles to one another in the plane of d_1, d_2, etc. Let the inclinations of d_1, d_2, etc.,



to Ox be a_1, a_2, etc. Then the displacements d_1, d_2, etc., have, in the direction of the x axis, components $d_1 \cos a_1$, $d_2 \cos a_2$, etc., and, in the direction of the y axis, components $d_1 \sin a_1$, $d_2 \sin a_2$, etc. Hence, 86 (III.), we have in the direction of the x axis a resultant displacement equal to $d_1 \cos a_1 + d_2 \cos a_2 +$ etc., which may be written $\Sigma d \cos a$; and, in the direction of the y axis, a resultant displacement equal to $d_1 \sin a_1 + d_2 \sin a_2 +$ etc., which may be written $\Sigma d \sin a$. Ox and Oy being at right angles, the resultant of these resultants has the magnitude, 86 (V.),

$$R = [(\Sigma d \cos a)^2 + (\Sigma d \sin a)^2]^{\frac{1}{2}},$$

the positive value being obviously the proper one to use.

The inclination θ of this resultant to the x axis is determined by the equation

$$\tan \theta = (\Sigma d \sin a)/(\Sigma d \cos a),$$

the signs of $\Sigma d \sin a$ and $\Sigma d \cos a$ determining which of the two directions indicated by this equation is that of R.

In adding together the components of the given displacements in the x and y axes respectively, we have assumed that the displacements are all in such directions as to give components in the directions of Ox, Oy respectively. If the directions of any are such as to give components in the directions xO or yO respectively, they must be (86, IV.) considered as negative in deter-

mining the resultant displacements in these axes. Thus $\Sigma d \cos \alpha$, $\Sigma d \sin \alpha$ are short expressions for the algebraic sums of all components of the form $d \cos \alpha$, $d \sin \alpha$ respectively.

Secondly, let the given components have any directions whatever.—Take three rectangular axes, Ox, Oy, Oz, and let the inclinations of the displacements d_1, d_2, etc., to the x, y, z axes, be $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, etc., respectively. Then the components of d_1, d_2, etc., in the direction of the x axis, are $d_1 \cos \alpha_1$, $d_2 \cos \alpha_2$, etc., and their resultant is $\Sigma d \cos \alpha$. Similarly, the resultants of the component displacements in the y and z axes are $\Sigma d \cos \beta$, $\Sigma d \cos \gamma$ respectively. Hence (88), if R is the magnitude of the resultant,

$$R = \{(\Sigma d \cos \alpha)^2 + (\Sigma d \cos \beta)^2 + (\Sigma d \cos \gamma)^2\}^{\frac{1}{2}}.$$

Also, if the direction cosines of the resultant with reference to the x, y, z axes, are λ, μ, ν respectively, we have

$$\lambda = (\Sigma d \cos \alpha)/R; \quad \mu = (\Sigma d \cos \beta)/R; \quad \nu = (\Sigma d \cos \gamma)/R.$$

91. *Examples.*

(1) $ABCD$ is a quadrilateral. Show that, if AC is produced to E, and CE made equal to AC, the resultant of component displacements represented by AC, DB, AD, and BC will be represented by AE.

(2) $ABCD$ is a parallelogram. E is the middle point of AB. Find the components, in the directions of AB and AD, of a displacement which has the direction and half the magnitude of the resultant of component displacements represented by AC and AD.

Ans. AE and AD.

(3) The resultant of two equal displacements of magnitude a, and inclined 60°, is equal to that of a and $2a$ inclined 120°.

(4) Two component displacements are represented by two chords of a circle drawn from a point P in its circumference and perpendicular to one another. Show that the resultant is represented by the diameter through the point.

(5) POP_1 and QOQ_1 are two perpendicular chords of a circle, whose centre is C. Show that the resultant of four component displacements represented by OP, OP_1, OQ, OQ_1, has the direction of OC and twice its magnitude.

(6) The resultant D of two displacements, d_1 and d_2, is perpendicular to d_2. Find the resultant of displacements $d_1/2$ and d_2, their inclination being the same as that of d_1 and d_2.

Ans. $d_1/2$ in a direction inclined $\tan^{-1}(D/d_2)$ to that of d_2.

(7) A point undergoes two component displacements, 60 ft. W. 30° S., and 30 ft. N. Find the resultant.

Ans. 51·9... ft. W.

(8) Show that three component displacements whose magnitudes are 1, 2, 3, and whose directions are represented by the sides of an equilateral triangle, taken the same way round, have a resultant whose magnitude is $\sqrt{3}$.

(9) A point undergoes three component displacements, 40 yds. N. 60° E., 50 yds. S., and 60 yds. W. 30° N. Find the resultant.

Ans. $10\sqrt{3}$ yds. W.

(10) A ship is carried by wind 4 mls. N., by her screw 8 mls. N. 15° W., and by a current 3 mls. E. 15° N. Find her resultant displacement in a north-easterly direction.

Ans. 9·4... mls.

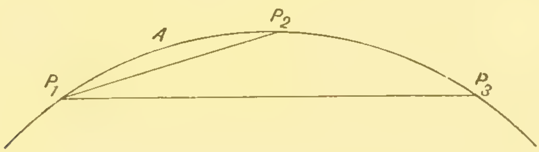
(11) A boat is headed directly across a river flowing from north to south, and reaches a point from which the starting point is found to bear N. 30° W., and is known to be at a distance of 400 ft. How far has the boat been carried by the current, and what distance would it have made in still water?

Ans. 346·4... and 200 ft. respectively.

(12) To an observer in a balloon his starting point bears N. 20° E., and is depressed 30° below the horizontal plane; while a place known to be on the same level as the starting point and 10 mls. from it, is seen to be vertically below him. Find the component displacements of the balloon in southerly, westerly, and upward directions.

Ans. 9·39..., 3·42..., and 5·77... mls. respectively.

92. *Velocity.*—The *mean velocity** of a moving point during a given time, is a quantity whose direction is that of the displacement produced during the time, and whose magnitude is the quotient of the magnitude of the displacement by the time. Thus, if a point move in the path A from P_1 to P_2 in the time t, its displacement in



that time is the straight line P_1P_2, the direction of its mean velocity is the direction of P_1P_2, and the magnitude of its mean velocity is P_1P_2/t.

In general the mean velocity of a point varies with the interval of time to which it applies. Thus, if in a time t' the point moves from P_1 to P_3, the direction of its mean velocity during t' is that of P_1P_3, and the magnitude is P_1P_3/t'. In the special case in which a point moves so that its mean velocity changes neither in magnitude nor in direction, it is said to move with uniform velocity. In that case its path must be a straight line. For, wherever P_2 and P_3 may be, P_1P_2 and P_1P_3 must have the same direction. It must also obviously be moving with uniform speed.

It will be seen that a point whose speed is uniform has not necessarily a uniform velocity. The speed is uniform if

$$\text{arc } P_1P_2/t = \text{arc } P_1P_3/t'.$$

But the velocity is not uniform, unless the chords P_1P_2 and P_1P_3 have the same direction, and their quotients by t and t' respectively, the same magnitude.

* The term *mean velocity* is employed by most writers to denote what we have called (42) *mean speed*.

93. The *instantaneous velocity* at a given instant (usually called velocity simply) is a quantity whose magnitude and direction are the limiting magnitude and direction of the mean velocity between that instant and another when the interval of time between them is made indefinitely small.

As bodies are found to require in all cases a finite time to traverse a finite distance, the instantaneous velocity of a body has always a finite value.

When the interval of time t (92) is made indefinitely small, P_2 is indefinitely near P_1, and the chord P_1P_2 coincides with the arc P_1P_2. Hence the direction of the instantaneous velocity at a given instant is that of the tangent to the path at the point occupied by the moving point at that instant; and its magnitude is equal to the instantaneous speed (43) of the point at that instant.

Velocity, having both magnitude and direction, is thus, like displacement, a vector (70).

94. *Measurement of Velocity.*—The specification of a velocity involves specification of both magnitude and direction. The direction may be described in terms of the unit of plane angle (21). The magnitude, being the quotient of a distance by an interval of time, is a quantity of the same kind as a speed (42), and may therefore be measured in terms of the unit of speed (45). A unit of speed is thus also a unit of velocity; and the results of 47–49 apply to units of velocity as well as to units of speed.

95. *Examples.*

(1) A point (see 91, Ex. 7) moves in a straight line from A to B, 60 ft., W. 30° S., in 10 sec., and thence in a straight line to C, 30 ft. N., in 20 sec. Find (a) the mean speed, and (b) the mean velocity during the whole time.

Ans. (a) 3 ft. per sec.; (b) 1·73... ft. per sec., W.

(2) A point moving with uniform speed in a circular path passes from one end of a diameter to the other in 10 sec. The radius being 30 cm., find (*a*) the mean speed, (*b*) the mean velocity, and (*c*) the instantaneous velocity at any instant.

Ans. (*a*) 9·4... cm. per sec.; (*b*) 6 cm. per sec. in the direction of the given diameter; (*c*) 9·4... cm. per sec. in the direction of the tangent at the point occupied by the moving point at the chosen instant.

(3) A man 6 ft. in height is walking at the rate of 4 mls. per hour directly away from a lamp-post 10 ft. high. Find the magnitude of the velocity of the extremity of his shadow.

Ans. 10 mls. per hour.

96. *Change of the Point of Reference.*—Velocity, being defined in terms of displacement, can be specified only by reference to some chosen point, which point of reference it is frequently desirable to change.

Since the direction and magnitude of a velocity are the direction and magnitude of a displacement, viz., either one which actually occurs in a unit of time, or one which would occur in that time were the velocity not variable, the propositions established in 71–74 for displacements apply also to velocities. Hence,

(1) The velocity of one point relative to another is equal and opposite to the velocity of the second relative to the first.

(2) If two sides of a triangle, taken the same way round, represent the velocities of P relative to Q, and of Q relative to O respectively, the third side, taken the opposite way round, will represent the velocity of P relative to O.

(3) If two sides of a triangle, taken the same way round, represent, one the velocity of P relative to O, and the other a velocity equal and opposite to that

of Q relative to O, the third side, taken the opposite way round, will represent the velocity of P relative to Q.

In the special case in which the velocities of P and Q relative to O have the same direction, the velocity of P relative to Q will be equal to the difference of those of P relative to O and of Q relative to O.

97. *Examples.*

(1) A is moving with velocity V in a north-easterly direction, B with an equal velocity in a direction 15° east of south. Show that A's velocity relative to B has a magnitude $V\sqrt{3}$ and is in a direction N. 15° E.

(2) Two points are moving with equal uniform speed v, the one in a circle of radius r, the other in a tangent to the circle. Both start at the same instant in the same direction from the point of contact of their paths. Find their relative velocity after t units of time.

Ans. $2v \sin \frac{vt}{2r}$, in a direction inclined to the tangent at an angle $\frac{1}{2}\left(\pi - \frac{vt}{r}\right)$.

(3) One railway train is running at 20 mls. per hour in a northerly direction. Another running at half the speed appears to a passenger in the former to be running at 25 mls. per hour. Find the direction of the velocity of the latter.

Ans. 71° 47′·4 W. or E. of N.

(4) To a person travelling at 8 mls. an hour along a road tending west, the wind appeared to come from the N.W. On his standing still, it seemed to shift 5° to the north. Find its velocity.

Ans. 64·9 mls. per hour N. 40° W.

(5) A man walks at the rate of 4 mls. per hour in a shower of rain. If the drops fall vertically with a speed of 200 ft. per sec., in what direction will they seem to him to fall?

Ans. In a direction inclined 1° 40′·8... to the vertical.

(6) Two candles, A and B, each 1 ft. long and requiring 4 and 6 hours respectively to burn out, stand vertically at a distance of 1 ft. The shadow of B falls on a vertical wall at a distance of 10 ft. from B. Find the speed of the end of the shadow.

Ans. 8 inches per hour.

(7) Two equal circles touch each other. Two moving points start in opposite directions from the point of contact and move on the circles with equal uniform speeds. Prove that the path of each, relative to the other, will be a circle whose radius is equal to the diameter of either of the first circles.

98. *Composition of Velocities.*—A point has two or more component velocities: it is required to find its resultant velocity.

As in 96, it may be shown that the propositions proved in 78 to be applicable to displacements are applicable also to velocities. Hence

(1) If two sides of a triangle, taken the same way round, represent two component velocities, the third side, taken the opposite way round, will represent the resultant velocity. This proposition is known as the *triangle of velocities.*

(2) If two component velocities be represented by two adjacent sides of a parallelogram taken opposite ways round, the diagonal of the parallelogram through their point of intersection will represent their resultant. This proposition is known as the *parallelogram of velocities.*

(3) If any number of component velocities be represented by $n-1$ of the sides of a polygon, taken the same way round, their resultant will be represented by the n^{th} side, taken the opposite way round. This proposition is known as the *polygon of velocities.*

99. *Resolution of Velocities.*—It follows also that velocities may be resolved into components in the same manner as displacements (see 79-84).

100. From the above propositions (98) there may be deduced trigonometrical and analytical expressions for the magnitude and direction of a resultant velocity in terms of the magnitudes and inclinations of the components, just as in the case of displacements. All the formulae of 85–90 hold if we take d_1 and d_2 to represent component velocities and R to represent the resultant velocity.

101. In the important case in which the position of a moving point is specified by reference to fixed rectangular axes, Ox, Oy, Oz, the components of the instantaneous velocity of the moving point in the directions of the x, y, and z axes are (93) equal to the rates of increase of the x, y, and z co-ordinates. They are thus denoted by $\dot{x}$, $\dot{y}$, $\dot{z}$.

102. *Examples.*

(1) A point has three component velocities, A, B, and C, in one plane. Their magnitudes are 4, 5, and 6 respectively, and their directions are such that A is inclined 30° to B, and C 60° to B and 90° to A. Find (a) the resultant of A and B, (b) the resultant of all three, and (c) the component of the resultant in the direction of B.

Ans. (a) $(41+20\sqrt{3})^{\frac{1}{2}}$, inclined to A at $\sin^{-1}\dfrac{5}{2(41+20\sqrt{3})^{\frac{1}{2}}}$;

(b) $(107+20\sqrt{3})^{\frac{1}{2}}$, inclined to C at $\tan^{-1}\dfrac{8+5\sqrt{3}}{17}$; ($c$) $8+2\sqrt{3}$.

(2) A boat's crew row $3\frac{1}{2}$ mls. down a river and back again in 1 hour 40 min. If the river have a current of 2 mls. per hour, find the rate at which the crew would row in still water.

Ans. 5 mls. per hour.

(3) A river 1 ml. broad is running at the rate of 4 mls. per hour; and a steamer which can make 8 mls. per hour in still water is to go straight across. In what direction must she be steered?

Ans. At an angle of 60° to the river bank.

(4) A ship has a north-easterly velocity of 12 knots (nautical

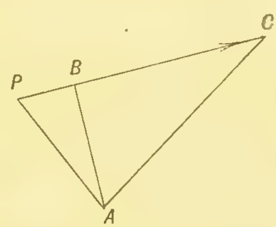
miles per hour). Find the magnitude of her velocity (*a*) in an easterly direction, (*b*) in a direction 15° W. of N.

Ans. (*a*) $6\sqrt{2}$, and (*b*) 6, knots.

(5) From a ship steaming east at 10 mls. an hour a shot is to be fired so as to strike an object which bears N.E. If the gun, properly elevated, can give the shot a horizontal velocity of 88 ft. per sec., towards what point of the compass must it be directed?

Ans. N. 38° 13′·9... E.

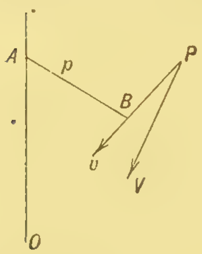
103. *Moment of a Velocity.*—The moment of the velocity of a moving point about a given fixed point (24) is the product of the magnitude of the velocity into the perpendicular from the given point on a line through the position of the moving point at the instant under consideration, and in the direction of its velocity.—Let P be the position of the moving point at the instant under consideration, A that of the fixed point. Let PC be the direction of the velocity, and v its magnitude. Let p be the length of the perpendicular AB from A on PC. Then the moment of v about A is pv. If PC represents the velocity in magnitude as well as direction, the magnitude of the moment of the velocity is evidently represented by twice the area of the triangle PAC.



If the moving point have a velocity represented by CP instead of PC, the moment of its velocity about A will be of the same magnitude. To distinguish between the equal moments of velocities represented by PC and by CP, they are considered to be of opposite sign. If the motion of the moving point is such that the radius vector AP moves counter-clockwise (*i.e.*, in the opposite direction to that of the hands of a clock), the moment of its velocity is considered to be positive. If its motion is such that the radius vector moves clockwise, the moment

of its velocity is considered to be negative. Thus the moment of the velocity PC is $-pv$; that of the velocity CP would be $+pv$.

104. The moment of the velocity of a moving point about a given line or axis, fixed in space (24), is the moment of the component of the velocity in a plane perpendicular to the given line, about the point of intersection of that plane with the given line. If P is the position of the moving point at the instant under consideration, V its velocity, OA the given line, v the component of V in a plane perpendicular to OA, A the intersection of OA with that plane, and AB (length $=p$) the perpendicular from A on the line through P representing v, the moment of V about OA is the product pv. The same convention of signs is employed as in 103.



105. The algebraic sum of the moments of two component velocities about any fixed point in their plane is equal to the moment of their resultant about the same point.—Let OA, OB be two component velocities whose resultant is OC, and P any point in their plane, either (Fig. 1) outside or (Fig. 2) inside the angle between the resultant and either of the components. Then, by a familiar geometrical proposition,* the sum of the areas of the tri-

* If a point P be taken in the plane of a parallelogram $OACB$, and lines drawn from it to the angular points, the area of the triangle OCP is equal to the sum or the difference of the areas of the triangles OAP, OBP, according as these triangles are on the same side or on opposite sides of OP. For

$$\text{area } OPC = \text{area } OAC + \text{area } APC \pm \text{area } OAP;$$

and, since the base OB of the triangle OBP is equal and parallel to the base AC of the triangles OAC and APC, and the altitude of OBP equal to the sum of the altitudes of OAC and APC,

$$\text{area } OBP = \text{area } OAC + \text{area } APC.$$

$$\text{area } OPC = \text{area } OBP \pm \text{area } OAP.$$

angles OAP and OBP in Fig. 1, and their difference in Fig. 2, is equal to the triangle OPC. But these triangles are proportional to the moments of the velocities OA, OB, OC

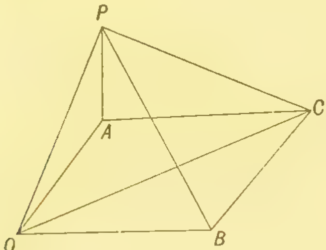


Fig. 1

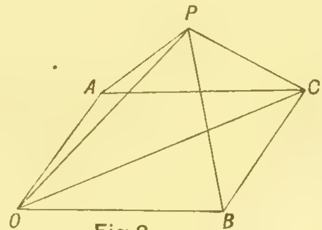


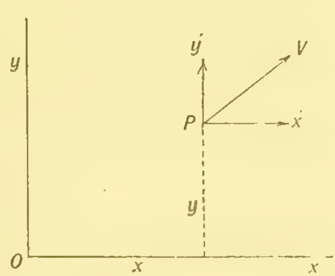
Fig. 2

respectively. And the moments of OA and OB have, in the case of Fig. 1, the same sign, and in that of Fig. 2, opposite signs. Hence the algebraic sum of the moments of OA and OB is equal to the moment of OC.

The cases in which the point P is on the line OA (or OB) or on the line OC, may be left to the reader. In the former, the moment of the one component is zero, and that of the other is equal to the moment of the resultant. In the latter, the moment of the resultant is zero, and the moment of the one component is equal and opposite to that of the other.

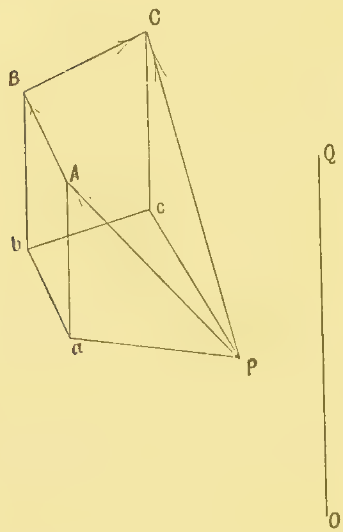
This proposition may obviously be extended to any number of component velocities in one plane.

106. If the position of the moving point P is specified by reference to fixed rectangular axes, Ox, Oy, in the same plane with P's velocity, its co-ordinates being x, y, its component velocities in the directions of the axes are $\dot{x}$ and $\dot{y}$ respectively, and their distances from O, y and x respectively. Hence the moments of the components about O are (103) $-\dot{x}y$ and $+\dot{y}x$ respectively. The moment of V



(the velocity of P) about O is therefore (105) $\dot{y}x-\dot{x}y$. If V is not in the xy plane, $\dot{y}x-\dot{x}y$ is obviously equal to its moment about the axis of z.

107. The algebraic sum of the moments of any number of component velocities about any fixed axis is equal to the moment of their resultant about the same axis.—Let the component velocities of the point P be represented by PA, AB, BC, and its resultant velocity therefore by PC.—Let a, b, c be the feet of perpendiculars from A, B, C on the plane through P perpendicular to the given fixed axis OQ. Then Pa, ab, bc, Pc are the components of PA, AB, BC, PC in this plane. Since Pa, ab, bc are in a plane perpendicular to the axis, the moment of Pc about the axis is (105) equal to the algebraic sum of the moments of Pa, ab, and bc. And since Pa, ab, bc, Pc are the components, in the aforementioned plane, of PA, AB, BC, PC respectively, the moments of the former about the given axis are equal respectively (104) to the moments of the latter. Hence the moment of PC about the given axis is equal to the algebraic sum of the moments of PA, AB, and BC.



108. *Examples.*

(1) AB is a diameter of a circle of which BC is a chord. When is the moment about A of a velocity represented by BC the greatest?

Ans. When angle $ABC=45°$.

(2) A moving point P has two component velocities, one of which is double the other. The moment of the smaller about a point O

in their plane is double that of the greater. Find the magnitude and direction of the resultant.

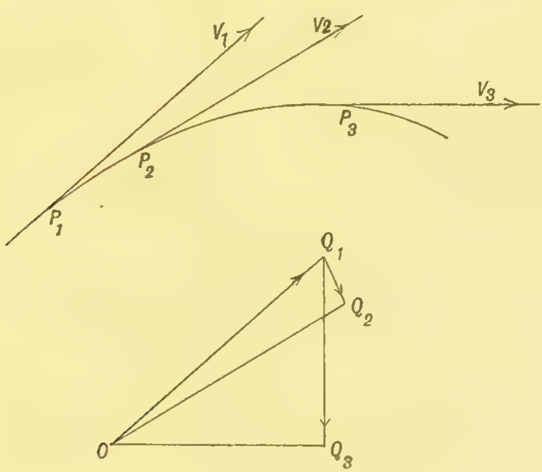
Ans. If α is the inclination of the greater component to PO, the resultant is $(5+4\cos\alpha(1-16\sin^2\alpha)^{\frac{1}{2}}+16\sin^2\alpha)^{\frac{1}{2}}$ times the smaller component, and is inclined to PO at the angle

$$\tan^{-1}\frac{6\sin\alpha}{2\cos\alpha+(1-16\sin^2\alpha)^{\frac{1}{2}}}.$$

(3) If the component velocities of a moving point can be represented by the sides of a plane polygon, taken the same way round, the algebraic sum of their moments about any point in their plane is zero.

(4) Show that, if the algebraic sums of the moments of the component velocities of a moving point about two points P and Q be each zero, the algebraic sum of their moments about any point in the line PQ will also be zero.

109. *Change of Velocity.*—The velocity of a moving point in general changes from instant to instant both in magnitude and in direction. Let $P_1P_2P_3$ be the path of



a point, and let V_1, which touches the path at P_1, represent the velocity of the point at P_1; and let V_2, V_3

similarly represent the velocities of the point at P_2 and P_3 respectively.

The change in the point's velocity, which has occurred in the time occupied by the point in moving from P_1 to P_2, is that velocity which must be compounded with the initial velocity V_1 to produce the final velocity V_2. Take any point O; from it draw OQ_1 and OQ_2, equal to and codirectional with V_1 and V_2. Join Q_1Q_2. Then (98) the final velocity OQ_2 is the resultant of the two components OQ_1 the initial velocity, and Q_1Q_2. Hence, Q_1Q_2 represents the change of velocity which the point has experienced between P_1 and P_2.

The change of velocity must be carefully distinguished from the change of speed. The change of speed in the above figure is $V_2 - V_1$ and is represented by $OQ_2 - OQ_1$.

110. *Acceleration.*—The *integral acceleration* during a given time is the change of velocity undergone by the moving point during that time.

The *mean acceleration* during any time is a quantity whose magnitude is the quotient of the integral acceleration by the time, and whose direction is that of the integral acceleration. Thus, if t units of time are occupied by the point in moving from P_1 to P_2 (109), the mean acceleration during that time is in the direction of Q_1Q_2 and of the magnitude Q_1Q_2/t. If $t+t'$ units of time are occupied in moving from P_1 to P_3, and if OQ_3 is drawn equal to V_3 and in the same direction, then the mean acceleration during these $t+t'$ units of time is in the direction Q_1Q_3 and of the magnitude $Q_1Q_3/(t+t')$.

Thus the mean acceleration of a point varies in general both in magnitude and direction with the interval of time to which it applies. In the special case in which it varies neither in magnitude nor in direction, the point is said to move with uniform acceleration.

The *instantaneous acceleration* of a moving point at a given instant (called usually the acceleration simply) is a quantity whose magnitude and direction are the limiting magnitude and direction of the mean acceleration between that instant and another when the interval of time between them is made indefinitely small. As (295) in the case of a body a finite time is required for a finite change of velocity, the instantaneous acceleration of a body can never have an infinite value.

If the point is moving with uniform acceleration, the instantaneous acceleration at any instant has clearly the same magnitude and direction as the mean acceleration for any interval.

Acceleration, having both magnitude and direction, is a vector (70), like displacement and velocity.

111. *Measurement of Acceleration.*—The specification of an acceleration involves specification both of its magnitude and of its direction. Its direction may be described in terms of the unit of angle (21). Its magnitude being the quotient of the magnitude of a certain velocity by an interval of time, is a quantity of the same kind as a rate of change of speed (52), and may therefore be measured in terms of the unit of rate of change of speed (56). This unit is thus called also the unit of acceleration; and the results of 57, 58 hold for units of acceleration.

112. *Examples.*

(1) The initial and final velocities of a moving point during an interval of 2 hours are 8 mls. per hour E. 30° N., and 4 mls. per hour N. Find (*a*) the integral, and (*b*) the mean acceleration.

Ans. (*a*) $4\sqrt{3}$ mls. per hour, W.; (*b*) $2\sqrt{3}$ mls.-per-hour per hour, W.

(2) A point moves in a horizontal circle with uniform speed v, starting from the north point and moving eastwards. Find the

integral acceleration when it has moved through (*a*) a quadrant, (*b*) a semicircle, (*c*) three quadrants.

Ans. (*a*) $v\sqrt{2}$, S.W.; (*b*) $2v$, W.; (*c*) $v\sqrt{2}$, N.W.

(3) The velocity v of a point moving in a straight line being supposed to vary as the square root of its distance s from a fixed point in the line, show that its instantaneous acceleration in any position is equal to $v^2/2s$.

(4) The velocity of a point moving in a straight line varies as the square root of the product of its distances from two fixed points in the line, show that its instantaneous acceleration varies as the mean of its distances from the fixed points.

113. *The Hodograph.*—The variation of the velocity of a moving particle from one position to another of its path may be studied by means of an auxiliary curve, called the hodograph of the path.

The velocity of a particle must have (295) indefinitely nearly the same magnitude and direction at points of its path which are indefinitely near. If therefore (109) the angle between OQ_1 and OQ_2 is indefinitely small, the length of OQ_1 must be indefinitely nearly equal to that of OQ_2. Hence the locus of the end Q of the line OQ, which represents the velocity of the moving particle P in its successive positions, must be a curve of continuous curvature. This curve is called the hodograph of the path. The point O is called the pole of the hodograph.

The hodograph has two important properties which may be proved as follows:—The straight line Q_1Q_2 represents in magnitude and direction the integral acceleration during the time τ occupied by P in moving from P_1 to P_2, and Q_1Q_2/τ represents the magnitude of the mean acceleration during the same time. When P_2 is taken indefinitely near P_1, the direction of Q_1Q_2 is the direction, and the magnitude of Q_1Q_2/τ is the magnitude, of the acceleration of P at the instant at which it is at

P_1. But when P_2 is taken indefinitely near P_1, and therefore Q_2 indefinitely near Q_1, the direction of Q_1Q_2 is that of the tangent to the hodograph at Q_1, and the magnitude of Q_1Q_2/τ is that of the velocity at Q_1 of the point Q in the hodograph. Hence (1) the direction of the acceleration of the moving point P at the instant at which it occupies a given position in its path is that of the tangent to the hodograph at the corresponding position of Q, and (2) the magnitude of the acceleration of P at the instant at which it occupies a given position in its path is equal to the magnitude of the velocity of Q at the corresponding position in the hodograph.

114. *Examples.*

(1) Show that the hodograph of a point moving with uniform speed in a straight path reduces to a point.

(2) A point moves with uniform acceleration, either in a straight or in a curved path. Show that the hodograph of the path is a straight line, and that the point in the hodograph moves with uniform speed.

(3) The hodograph of a point which moves with uniform speed in a circle, is a circle, in which the corresponding point moves also with uniform speed.

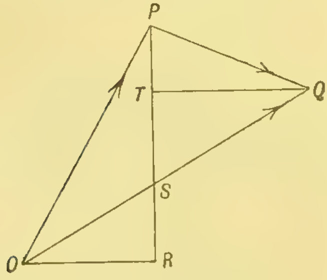
(4) If a point move in either a parabola, an ellipse, or an hyperbola, so that the moment of its velocity about a focus is constant, the hodograph is a circle. [Note that the locus of the foot of the perpendicular from a focus on a tangent is a circle in the case of either the ellipse or the hyperbola, and in the case of the parabola a straight line. Note also that the locus of the foot of the perpendicular from the vertex of a parabola on a straight line drawn through the focus is a circle.]

(5) The hodograph of a point moving in an ellipse so that the moment of its velocity about the centre is constant, is a similar ellipse. [Note that the area of the parallelogram formed by drawing tangents to an ellipse at the extremities of a pair of conjugate diameters is constant.]

115. *Change of the Point of Reference.*—Acceleration being simply the velocity which must be compounded with the velocity of a moving point at a given instant, to produce the velocity which it either has after unit of time, or would have if the acceleration were uniform, the propositions (96) dealing with the change of the point of reference in the case of velocities, apply also in the case of accelerations.

116. *Composition and Resolution of Accelerations.*—Similarly the laws of the composition of velocities (98) may be shown to be those according to which accelerations also are compounded. We have thus propositions called the *triangle*, the *parallelogram*, and the *polygon of accelerations* identical in form with the corresponding propositions for velocities. Hence also accelerations may be resolved after the same manner as velocities (99); and the formulae of 85–90 hold, if d_1, d_2, etc., and R denote component and resultant accelerations.

117. The component of an acceleration in any direction is equal to the rate of increase of the component in that direction of the velocity.—Let OP and OQ be the initial and final velocities of a moving point during a given time. Then PQ is the integral acceleration. Drawing OR, QT at right angles to any line PR, we have $PT = RT - RP$. If then τ be the time,



$$PT/\tau = (RT - RP)/\tau.$$

Now PT, RP, and RT are the components in the line PR of PQ, OP, and OQ respectively. If τ is indefinitely short, PT/τ is thus the component of the instantaneous acceleration in the direction PR, and $(RT - RP)/\tau$ is the instantaneous rate of increase of the component velocity in the same direction.

118. If the position of a moving point be specified by reference to rectangular axes Ox, Oy, Oz, its component accelerations in their directions will therefore be equal to the rates of change of its component velocities in their directions, namely, of $\dot{x}$, $\dot{y}$, $\dot{z}$ respectively. They are therefore (55) $\ddot{x}, \ddot{y}, \ddot{z}$.

119. *Examples.*

(1) A ball is let fall in an elevator which is rising with an upward acceleration of 7·2 kilometres-per-min. per min. The acceleration of the falling ball relative to the earth is 981 cm.-sec. units. Find its acceleration relative to the elevator.

Ans. 1,181 cm.-sec. units towards the floor.

(2) Two railway trains are moving in directions inclined 60°. The one A is increasing its speed at the rate of 4 ft.-per-min. per min. The other B has the brakes on and is losing speed at the rate of 8 ft.-per-min. per min. Find the relative acceleration.

Ans. $4\sqrt{7}$ ft.-min. units, inclined $\sin^{-1}\sqrt{\frac{3}{7}}$ to the direction of motion of A, and $\frac{\pi}{3}-\sin^{-1}\sqrt{\frac{3}{7}}$ to that of B.

(3) The locus of the extremity of the straight line representing either of the two equal components of a given acceleration, is a straight line perpendicular to the straight line representing the given acceleration and through its middle point.

(4) A bullet is fired in a direction towards a second bullet which is let fall at the same instant. Prove that the line joining them will move parallel to itself and that the bullets will meet.

(5) Find the resultant of four component accelerations, represented by lines drawn from any point P within a parallelogram to the angular points.

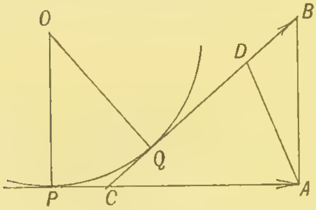
Ans. If C is the point of intersection of the diagonals, PC represents the direction of the resultant, and $4PC$ its magnitude.

(6) The resultant of two accelerations a and a' at right angles

to one another is R. If a be increased by 9 units and a' by 5, the magnitude of R becomes increased to three times its former value, and its direction becomes inclined to a at the angle of its former inclination to a'. Find a, a', and R.

Ans. 3, 4, and 5 units respectively.

120. *Tangential and Normal Acceleration.*—An important special case of the resolution of accelerations is the resolution of the acceleration of a point moving in a plane curve, into components in and normal to the direction of motion at any instant.—Let P, Q be points on the path, and PA, QB tangents to the path at P, Q respectively. Let CA, CB represent the velocities of the moving point at P and Q respectively. From CB cut off CD equal to CA. If the point Q be made to approach P, the angle BCA becomes ultimately zero, and the angles CDA and CAD therefore ultimately right angles. Now AB represents the integral acceleration between P and Q, and it may be resolved into AD and DB as components. Hence ultimately AD and DB, divided by the time, represent the normal and tangential components of the instantaneous acceleration at P.



Since CD was made equal to CA, DB is the change of speed. Hence ultimately, when P and Q coincide, DB divided by the time is the rate of change of the speed of the moving point. Hence the tangential component of the acceleration of a moving point is equal to the rate of change of speed.

From P, Q draw PO, QO, normals to the path at those points, and let them meet in O. Then the angle QCA is equal to the angle QOP. Calling these angles θ, the velocity at P, V, and the component integral acceleration AD, v, we have

$$v = 2V\sin(\theta/2).$$

If a is the component in the direction AD of the mean acceleration between P and Q, and t the time of motion from P to Q, we have thus

$$a=\frac{v}{t}=\frac{2V\sin(\theta/2)}{t}.$$

Now ultimately the time of motion from P to Q may be put equal to PQ/V. Hence, calling PQ, s, we have

$$a=\frac{V^2}{s}2\sin(\theta/2).$$

Also ultimately PQ may be considered an arc of a circle, and PO, QO become equal to one another and to the radius of curvature (ρ) of the path at P, in which case $s=\rho\theta$. Hence

$$a=\frac{V^2}{\rho}\cdot\frac{2\sin(\theta/2)}{\theta}.$$

Also, θ being ultimately indefinitely small,

$$2\sin(\theta/2) \doteq \theta.$$

Hence $$a=V^2/\rho.$$

Now a, being the mean acceleration in the direction AD, becomes ultimately the instantaneous acceleration normal to the path at P. Hence the normal component of the acceleration of a point moving in a curved path is the product of the square of its velocity into the curvature of its path.

121. If the path be a circle, the radius of curvature is the radius of the circle. Also a normal to the circle through any point passes through the centre. Hence the acceleration of a point moving with uniform speed in a circle is directed towards the centre, and is equal to the quotient of the square of the speed by the radius.

If T be the time of a complete revolution (the periodic time) of the point in the circle, and if V be the uniform speed and R the radius, $V=2\pi R/T$. Hence the acceleration has the magnitude $4\pi^2 R/T^2$. (See also 131.)

122. *Examples.*

(1) A circus rider is moving with the uniform speed of a mile in 2 min. 40 sec. round a ring of 100 ft. radius. Find his acceleration towards the centre.

Ans. 10·89 ft.-sec. units.

(2) Show that a shot fired at the equator with either a westerly velocity of 8,370·7 metres per second, or an easterly velocity of 7,440·5 m. per sec., will, if unresisted, move horizontally round the earth, completing its circuit in about $1\frac{1}{3}$ or $1\frac{1}{2}$ hours respectively. [Data : The mean radius of the earth is 6,370,900 metres ; the speed of a point on the equator 465·1 m. per sec. ; and the acceleration of a falling body 9·81 m.-sec. units.]

(3) A point moving in a circular path, of radius 8 in., has at a given position a speed of 4 in. per sec., which is changing at the rate of 6 in.-per-sec. per sec. Find (a) the tangential acceleration ; (b) the normal acceleration ; (c) the resultant acceleration.

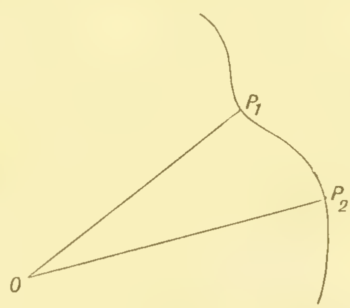
Ans. (a) 6 in.-sec. units ; (b) 2 in.-sec. units ; (c) $2\sqrt{10}$ in.-sec. units, the direction being inclined at $\tan^{-1} 3$ to the normal.

(4) If different points be describing different circles with uniform speeds and with accelerations proportional to the radii of their paths, their periodic times will be the same.

123. *The moment of an acceleration* is defined in exactly the same way as the moment of a velocity. See 103 and 104. Also the propositions of 105 and 107, being deductions from the parallelogram law, apply to accelerations as well as to velocities. And it may be shown, as in 106, that, if the position of a moving point be referred to rectangular axes of co-ordinates, the moment of its acceleration about the z axis is equal to $\ddot{y}x-\ddot{x}y$.

124. The moment of the acceleration of a moving point about a fixed point in the plane of its motion is equal to the rate of change of the moment of its velocity about the same point.—For the final velocity in any time is the resultant of the initial velocity and the integral acceleration. In an indefinitely short time, the position of the moving point does not appreciably change, and lines representing these vectors, drawn through the appropriate positions (103), ultimately intersect in one point. Hence (105) the moment of the final velocity is equal to the sum of the moments of the initial velocity and integral acceleration, and the moment of the integral acceleration to the difference of the moments of the initial and final velocities. Dividing by the time we reach the above result.

125. *Angular Displacement* of a Point.*—The angular displacement of a moving point about a given point in a given time, is the angle between the initial and final positions of the radius vector from the given point. Thus, if the point has moved from P_1 to P_2 its angular displacement, relative to O, is P_1OP_2.

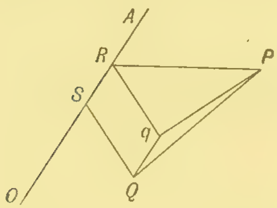


That an angular displacement about a given point may be completely specified, the magnitude of the angle must be given through which the radius vector has moved, the direction of the radius vector's motion, and the plane in which its initial and final positions lie. This plane is specified if a line normal to it be given; and the direction of the radius vector's motion is specified if this line is always so drawn from a point in the plane of the displacement that, on looking along it towards that plane,

* When there is danger of confounding angular displacements with the displacements considered in 69, the latter are called linear displacements.

the radius vector is seen to move counter-clockwise, *i.e.*, in a direction opposite to that of the hands of a clock. An angular displacement about a point may therefore be completely represented by a line normal to the plane of the displacement, whose direction is determined by the above convention, and whose length is proportional to the magnitude of the angular displacement. By the direction of an angular displacement is meant the direction of this line.

126. The angular displacement in a given time of a moving point about a given line or axis, is the inclination of perpendiculars from the initial and final positions of the moving point on the axis. Let OA be the given axis, P and Q the initial and final positions of the moving point, and PR and QS perpendiculars to OA. Then the inclination of PR to QS is the angular displacement about OA. Complete the rectangle RQ by the lines Rq, Qq. Then (8), since Rq is parallel to SQ, PRq is the angular displacement.



Since the plane of PqR is perpendicular to OA, and Qq being parallel to RS is perpendicular to that plane, Pq is the projection of PQ on that plane. Hence the angular displacement is the angle subtended by the projection of the linear displacement on a plane perpendicular to the axis, at the point of intersection of the axis with that plane.

127. *Angular Velocity of a Point.**—The *mean angular velocity* of a moving point about a given point, during a given time, is a quantity whose direction is that of the angular displacement during the time, and

* When there is danger of confounding the velocity of 92 with angular velocity, the former is called linear velocity.

whose magnitude is the quotient of the angular displacement by the time.

The mean angular velocity varies in general with the time. In cases in which it does not, the angular velocity is said to be uniform.

If the motion of the moving point is confined to a plane, its angular velocity must have one of two opposite directions. In other words, it can vary only as to magnitude and sign.

The *instantaneous angular velocity* of a point at a given instant has a magnitude and a direction which are the limiting magnitude and direction of the mean angular velocity between that instant and another, when the interval of time between them is made indefinitely small.

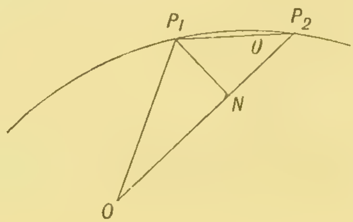
The *mean* and *instantaneous* angular velocities about a given axis are defined in a manner similar to that in which they are defined with reference to a given point.

128. *Measurement of Angular Velocity.*—The measure of an angular velocity being the quotient of the measure of a certain angle by that of a certain time, the most convenient unit of angular velocity will be unit of angle per unit of time. The unit of angle which is usually employed in measuring angular velocities is the radian. Unit of angular velocity in terms of the radian is one radian per unit of time.

As the magnitude of the radian is (21) independent of that of the unit of length, the magnitude of the radian per unit of time depends only upon that of the unit of time and is inversely proportional to it.

129. *Relation between Angular and Linear Velocity.*—Let the moving point P be displaced from P_1 to P_2 in the time t with the mean linear velocity v, and the

mean angular velocity ω about the point O. Then $\omega = P_1OP_2/t$, and the chord $P_1P_2 = vt$. From P_1 draw



P_1N perpendicular to OP_2. Then, if the angle P_1P_2N be called θ,

$$P_1N = P_1P_2 \sin\theta = vt \sin\theta.$$

Hence $$\sin P_1OP_2 = vt \sin\theta / OP_1.$$

If now P_2 be indefinitely near P_1, ω and v become instantaneous velocities at P_1, θ becomes the angle between the radius vector and the direction of the linear velocity at P_1, and $\sin P_1OP_2$ becomes equal to P_1OP_2. Hence, if r is the radius vector, $\omega = v \sin\theta/r$. Hence the angular velocity of a moving point about a given point, expressed in radians, is equal to the component of its linear velocity perpendicular to the radius vector from the given point, divided by the length of the radius vector.

130. If the point be moving in a circle, its linear velocity is at all points perpendicular to the radius vector from the centre. Hence, if r is the radius and ω the angular velocity about the centre, $\omega = v/r$.

131. Hence the normal component of the linear acceleration of a point moving in a circle, which (121) is equal to v^2/r, is, in terms of angular velocity about the centre, equal to $\omega^2 r$.

132. *Moment of Velocity in Terms of Angular Velocity.*—Since P_1P_2/t (129), when t is small, is the velocity of

the moving point, its moment about O is twice the area of the triangle OP_1P_2 divided by the time. Hence, if pv be written for the moment (103),

$$pv = OP_2 \,.\, P_1N/t = vr \sin\theta = \omega r^2.$$

133. *Areal Velocity.*—The area swept over by the radius vector of a moving point per unit of time, is sometimes called its areal velocity. It follows from 132 that the areal velocity of the point P (129) is represented by area OP_1P_2/t and is equal to $\frac{1}{2}\omega r^2$.

134. These results (129-133) apply also to angular velocities about axes, provided v stand for the component linear velocity in a plane perpendicular to the given axis, and r for the perpendicular distance of the point from the given axis.

135. *Angular Acceleration of a Point.*—We might define the angular acceleration of a moving point about a given point, as we did its angular velocity, generally. We restrict ourselves, however, to the useful case of the angular acceleration about a given axis. An angular velocity about a given axis must have one of two opposite directions, and can vary therefore in magnitude and sign only. Hence the integral angular acceleration about a given axis is the difference between the final and initial values of the angular velocity about that axis; the mean angular acceleration in a given time is equal to the integral acceleration divided by the time; and the magnitude of the instantaneous angular acceleration at a given instant is the limiting value of the mean angular acceleration between that instant and another when the interval of time between them is made indefinitely small, or in other words it is the rate of change of angular velocity.

The angular acceleration of a point moving in a plane about a given point in that plane is an angular acceleration about a given axis.

136. *Measurement of Angular Acceleration.*—The most convenient unit of angular acceleration is clearly unit of angular velocity per unit of time, *e.g.*, one radian-per-sec. per sec. With the radian as unit of angle its dimensions are $[T]^{-2}$.

137. *Examples.*

(1) The earth makes a complete rotation in 86,164 mean solar seconds. Assume her radius to be 6,370,900 metres, and find (*a*) the angular velocity, and (*b*) the linear velocity of any point on the equator.

Ans. (*a*) $\frac{1}{13713}$ radians per sec.; (*b*) 465·1 m. per sec.

(2) A wheel of a carriage which is travelling at the rate of 7 mls. per hour is 3 ft. in diameter. Find the angular velocity of any point of the wheel about the axle.

Ans. 6·8... rad. per second.

(3) Compare the angular velocities of the hour, minute, and second hands of a watch.

Ans. As 1 : 12 : 720.

(4) Express in terms of the radian per minute an angular velocity of 20° per second.

Ans. 20·94....

(5) A point is moving with uniform speed v in a circle of radius r. Show that its angular velocity about any point in the circumference is $v/2r$.

(6) The angular velocity of a point moving with uniform speed in a straight line is inversely proportional to the square of the distance of the point from a fixed point not in the line.

(7) Show that the angular velocity of the earth about the sun is proportional to the apparent area of the sun's disc. [Datum: The radius vector from the sun to the earth sweeps over equal areas in equal times.]

(8) If the velocity of a particle be resolved into several components in one plane, its angular velocity about any fixed point in the plane is the sum of the angular velocities due to the several components.

(9) A wheel makes 200 revolutions per hour. Express its angular velocity (*a*) in radians per sec., (*b*) in degrees per min.

Ans. (*a*) $\frac{\pi}{9}$; (*b*) 1,200.

(10) Reduce an angular acceleration of 300 radians-per-min. per min. (*a*) to revolution-hour units, (*b*) to degree-second units.

Ans. (*a*) $\frac{540,000}{\pi}$; (*b*) $\frac{15}{\pi}$.

(11) A point P moves in a parabola with a constant angular velocity about the focus S. Show that its linear velocity is proportional to $SP^{\frac{3}{2}}$.

CHAPTER IV.

TRANSLATION:—UNDER GIVEN ACCELERATIONS.

138. *Unconstrained Motion.*—The motion of a point under given accelerations will depend upon the degree of its freedom to move (35). We shall take, first, cases of unconstrained motion, the moving point having all three degrees of freedom.

Case I.—The Acceleration being Zero.—If there is no acceleration, there is no change in either the magnitude or the direction of the velocity. The path is therefore a straight line, and the magnitude of the velocity is constant. Hence the mean and instantaneous velocities, and therefore also the mean velocity and mean speed, have the same values (93 and 43). The results of 61 are thus at once applicable to this case.

139. *Examples.*

(1) A point moves with a uniform velocity of 2 cm. per sec. Find the distance from the starting point at the end of 1 hour.

Ans. 72 metres.

(2) Two trains having equal and opposite velocities, and consisting each of 12 carriages, of 23 ft. length, are observed to take 9 sec. to pass one another. Find the magnitude of their velocities.

Ans. 20·91 mls. per hour.

(3) Two points move with uniform velocities of 8 and 15 ft. per sec. in straight lines inclined 90°. At a given instant their distance is 10 ft., and their relative velocity is inclined 30° to the line joining them. Find (*a*) their distance when nearest, and (*b*) the time after the given instant at which their distance will be least.

Ans. (*a*) 5 ft.; (*b*) $\frac{5}{17}\sqrt{3}$ sec.

140. *Case II.—The Acceleration Uniform.*—The motion of a point under a uniform acceleration will be different according as the point has or has not at any instant a velocity inclined to the direction of its acceleration.

(*a*) *Rectilinear motion.*—If at any instant the velocity of the moving point is in the same straight line with the acceleration, the path is a straight line. For (109) OQ_1 and Q_1Q_2 being in the same straight line, so also are OQ_1 and OQ_2. Hence the velocity does not vary in direction. Also, OQ_1Q_2 being a straight line, $Q_1Q_2 = OQ_2 - OQ_1$. If then t is the time in which the velocity changes from OQ_1 to OQ_2,

$$Q_1Q_2/t = (OQ_2 - OQ_1)/t;$$

and t being taken indefinitely small, we find that the instantaneous acceleration is equal to the rate of change of the magnitude of the velocity, and therefore (93) to the rate of change of speed. Hence the results of 63–66 are applicable to this case.

We have a familiar instance of the motion under consideration in the motion of bodies under gravity upwards or downwards through short distances at the surface of the earth, except in so far as their velocity is modified by the resistance of the air. For all bodies falling freely near the surface of the earth are found to have a downward acceleration of about 32·2 ft.-sec. units, or 981 cm.-sec. units. When represented by a symbol, the

special symbol g* is usually employed to denote this acceleration.

141. *Examples.* †

(1) A body is projected vertically upwards with a velocity of 300 ft. per sec. Find (a) its velocity after 2 sec.; (b) its velocity after 15 sec.; (c) the time required for it to reach its greatest height; (d) the greatest height reached; (e) its displacement at the end of 15 sec.; (f) the space traversed by it (*i.e.* the length of path described) in the first 15 sec.; (g) its displacement when its velocity is 200 ft. per sec. upwards; (h) the time required for it to attain a displacement of 320 ft. [Note that if the upward direction be taken as positive, the acceleration in this case is negative.]

Ans. (a) 235·6 ft. per sec. upwards; (b) 183 ft. per sec. downwards; (c) 9·3... sec.; (d) 1,397·5 ft.; (e) 877·5 ft. upwards; (f) 1,917·5 ft.; (g) 776·3 ft. upwards; (h) 1·13 sec. in ascending, 17·5 sec. in descending.

(2) A ball is projected vertically upwards from a window half way up a tower 117·72 metres high, with a velocity of 39·24 m. per sec. After what times and with what speeds does it (a) pass the top of the tower ascending; (b) pass the same point descending; and (c) reach the foot of the tower?

Ans. (a) 2 sec., 19·62 m. per sec.; (b) 6 sec., 19·62 m. per sec.; (c) $(4+2\sqrt{7})$ sec., $19{\cdot}62 \times \sqrt{7}$ m. per sec.

(3) A stone is dropped into a well, and the splash is heard in 3·13 sec. Given that sound travels in air with a uniform velocity of 332 metres per sec., find the depth of the well.

Ans. About 44·1 m.

* The value of g at any place near the earth's surface is given approximately in centimetre-second units by the following formula, in which λ is the latitude of the place, and h its height above sea-level,

$$g = 980{\cdot}6056 - 2{\cdot}5028 \cos 2\lambda - {\cdot}000003h.$$

† In problems on falling bodies the resistance of the air is not to be taken into account. When the value of g is not given it is to be taken as 32·2 ft.-sec. units or 981 cm.-sec. units.

(4) Show that a body, projected vertically upwards, requires twice as long a time to return to its initial position as to reach the highest point of its path, and has, on returning to its initial position, a speed equal to its initial speed.

(5) A stone projected vertically upwards returns to its initial position in 6 sec. Find (*a*) its height at the end of the first second, and (*b*) what additional initial speed would have kept it 1 sec. longer in the air.

Ans. (*a*) 80·5 ft.; (*b*) 16·1 ft. per sec.

(6) A body let fall near the surface of a small planet is found to traverse 204 ft. in the fifth and sixth seconds. Find the acceleration.

Ans. 20·4 ft.-sec. units.

(7) A particle describes in the n^{th} second of its fall from rest a space equal to p times the space traversed in the $(n-1)^{\text{th}}$ second. Find the whole space described.

Ans. $(1-3p)^2 g/[8(1-p)^2]$.

(8) A body uniformly accelerated, and starting without initial velocity, passes over b feet in the first p seconds. Find the time of passing over the next b ft.

Ans. $p(\sqrt{2}-1)$ sec.

(9) A ball is dropped from the top of an elevator 4·905 metres high. Find the times in which it will reach the floor, (*a*) when the elevator is at rest; (*b*) when it is moving with a uniform downward acceleration of 9·81 m.-per-sec. per sec.; (*c*) when moving with a uniform downward acceleration of 4·905 m.-per-sec. per sec.; (*d*) when moving with a uniform upward acceleration of 4·905 m.-per-sec. per sec.

Ans. (*a*) 1 sec.; (*b*) ∞; (*c*) $\sqrt{2}$ sec.; (*d*) $\sqrt{\frac{2}{3}}$ sec.

(10) If s_1, s_2 are the heights to which a body can be projected with a given initial vertical velocity at two places on the earth's surface at which the accelerations of falling bodies are g_1 and g_2 respectively, show that $s_1 g_1 = s_2 g_2$.

(11) A stone A is let fall from the top of a tower 483 ft. high.

At the same instant another stone B is let fall from a window 161 ft. below the top. How long before A will B reach the ground?

Ans. $(\sqrt{6}-2)\sqrt{5}$ sec.

(12) A ball falling from the top of a tower had descended a ft. when another was let fall at a point b ft. below the top. Show that if they reach the ground together the height of the tower is $(a+b)^2/4a$ ft.

(13) If two bodies be projected vertically upwards with the same initial velocity V, at a interval of t sec., prove that they will meet at a height $\frac{g}{2}\left(\frac{V^2}{g^2}-\frac{t^2}{4}\right)$.

(14) Two stones are falling in the same vertical line. Show that if one can overtake the other, it will do so after the same lapse of time, even if gravity cease to act.

(15) Bodies are projected vertically downwards from heights h_1, h_2, h_3, with velocities v_1, v_2, v_3 respectively, and they all reach the ground at the same moment. Show that

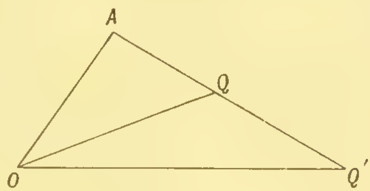
$$(h_1-h_2)/(v_1-v_2)=(h_2-h_3)/(v_2-v_3)=(h_3-h_1)/(v_3-v_1).$$

(16) Two points move in straight lines with uniform accelerations. Show that if at any instant their velocities are proportional to their respective accelerations, the path of either relative to the other will be rectilinear.

(17) Particles are projected vertically upwards from different points in a horizontal straight line AX, with velocities respectively proportional to the distances of the points of projection from A. Prove that all the particles when at their highest points will be on a parabola whose vertex is A.

142. (*b*) *Curvilinear motion.*—If the moving point has at any instant a velocity inclined to the direction of its acceleration, the direction of the velocity must change with the time, and consequently the path must be

a curved line. For if OA is the initial velocity, and AQ, AQ', the integral accelerations after t and t' units of time respectively, OQ and OQ' are the velocities after these intervals of time. And since AQ and AQ' have the same direction, OQ and OQ' must have different directions.



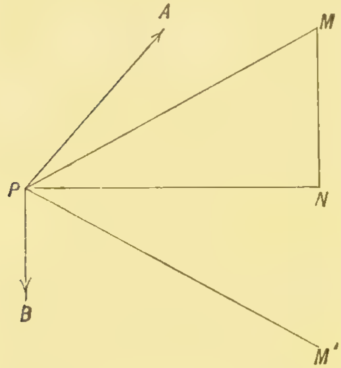
Nevertheless the component acceleration in any given direction being uniform in this case (140), the formulae of 63-66 apply to curvilinear as well as to rectilinear motion, provided we restrict our attention to a component motion in a given direction.

Curvilinear motion under uniform acceleration is of interest because it is the motion which *projectiles* near the earth's surface would have, if they were not resisted by the air, and if their accelerations were rigorously, as they are approximately, the same at all points of their paths.

143. *To find the Velocity after any Time.*—The moving point has after the time t two component velocities, one the initial velocity V, represented by OA (142), the other the integral acceleration at (if a is the acceleration), represented by AQ. If then the inclination of the two be given, the resultant, represented by OQ, may be found by 100.

144. *To find the Displacement of the Point after any Time.*—The moving point will have two component displacements after any time t, one of magnitude Vt, in the direction of the initial velocity, the other of magnitude $\frac{1}{2}at^2$, in the direction of the acceleration. The inclination of these displacements being known, their resultant may be determined by 85.

145. *To find the Displacement in any given Direction.**—Let P be the initial position of the moving point, PA the direction of the initial velocity V, and PB that of the acceleration a. (We draw PB vertical because of the importance of this problem in the study of projectiles.) Draw PN perpendicular to PB. Let PA be inclined to PN at the angle θ.† It is required to determine the displacement which the point will reach in the direction of PM, inclined to PN at the angle α.



The initial velocity has a component $V \sin(\theta - \alpha)$ perpendicular to PM. The acceleration has a component $a \cos \alpha$ perpendicular to PM and opposite in direction to the above component of the initial velocity. Hence, if t is the time at the end of which the displacement perpendicular to PM is zero, we have (64)

$$t = \frac{2V \sin(\theta - \alpha)}{a \cos \alpha}.$$

Now the point has in the direction PN a velocity $V \cos \theta$ and no acceleration. Hence in the time t its displacement in that direction is

$$\frac{2V \sin(\theta - \alpha)}{a \cos \alpha} \times V \cos \theta.$$

Let PN represent this displacement. Draw NM perpendicular to PN. Then PM is the required displacement

* In gunnery the displacement of a projectile in a given direction is called its range on a given plane; the time required to reach that displacement is called the time of flight.

† In gunnery this angle is called the elevation of the projectile.

in the direction PM. But $PM = PN/\cos\alpha$. Hence, denoting PM by R,

$$R = \frac{2V^2 \sin(\theta - \alpha)\cos\theta}{a\cos^2\alpha}.$$

Expanding $\sin(\theta - \alpha)$, adding and subtracting

$$V^2 \sin\alpha/(a\cos^2\alpha),$$

and remembering that $2\cos^2\theta - 1 = \cos 2\theta$, we find

$$R = \frac{V^2[\sin(2\theta - \alpha) - \sin\alpha]}{a\cos^2\alpha}.$$

The required displacement is therefore determined in terms of known quantities.

146. If the given direction be upon the other side of PN, viz., that of the line PM' (the angle $M'PN$ being equal to α), we obtain the result

$$R = \frac{V^2[\sin(2\theta + \alpha) + \sin\alpha]}{a\cos^2\alpha}.$$

If therefore in this case the inclination of the given direction to PN be considered negative, so that angle $M'PN = -\alpha$, we get the same expression for R as in 145.

147. If θ' is such that $2\theta' - \alpha = 180° - (2\theta - \alpha)$, we have

$$\sin(2\theta' - \alpha) = \sin(2\theta - \alpha).$$

Hence R will have the same value, whether the inclination of the initial velocity to PN be θ or $\theta' = 90° - \theta + \alpha$. With a given acceleration and an initial velocity of given magnitude, there are therefore two directions of initial velocity, and therefore two paths, by which the point may attain a given displacement in a given direction.

148. The above expression for R involves V, a, θ and α. If V, a, and α are given, θ is the only variable. The magnitude of R will therefore depend upon that of θ. Now $\sin(2\theta - \alpha)$ has its greatest value when $2\theta - \alpha = 90°$. Hence, with a given acceleration and an initial velocity

of given magnitude, the displacement in a given direction has its greatest value, viz., $V^2(1-\sin\alpha)/(a\cos^2\alpha)$, when $\theta=45°+\alpha/2$. When θ has this value, θ' has the same value. Hence there is but one direction of initial velocity by which the maximum displacement in the given direction can be attained; and that direction bisects the angle between the direction opposite to that of the acceleration, and the given direction.

As $\theta'=90°-\theta+\alpha$, we have

$$\theta'-(45°+\alpha/2)=45°+\alpha/2-\theta.$$

Hence the two directions of initial velocity of 147 are equally inclined to the direction for maximum displacement.

149. In the important special case in which PM coincides with PN, we have $\alpha=0$. Hence

$$R^*=V^2\sin 2\theta/a,$$

and this displacement is attained whether the inclination of PA to PN have the value θ or the value $90°-\theta$. The greatest possible value of R in this case is V^2/a, and it is attained when the inclination of the initial velocity has the value 45°.

150. The important practical problem of determining the direction of an initial velocity of given magnitude, with which the moving point will pass through a given point, may be solved at once by means of the above expression (145) for R. For the point, say M, being given, PM (*i.e.*, R) and α are known, and a and V being given, all the quantities involved in this expression except θ are known. We thus have for determining θ,

$$\theta=\tfrac{1}{2}\sin^{-1}\left(\frac{aR\cos^2\alpha}{V^2}+\sin\alpha\right)+\frac{\alpha}{2}.$$

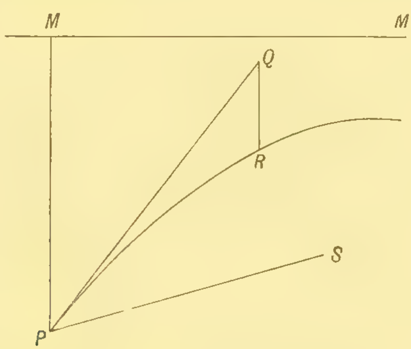
As pointed out above, θ may clearly have either of two

* In gunnery the range on a horizontal plane.—The reader should prove this special case directly.

values. In practice allowance must of course be made for the resistance of the air.

151. *To Determine the Path of the Point.*—The acceleration and initial velocity being given, the value of R in the above expression (145) will depend upon that of a. If different values be given to a, the displacements of the moving point in known directions, and therefore as many positions as we please of the point in its path may be determined. Thus, as the reader who is familiar with analytical geometry will see, this expression is an equation to the path of the moving point expressed in polar co-ordinates.

152. The path may be determined also by the following geometrical method. Let P be the initial position of



the moving point, and let PQ represent in direction the direction of the initial velocity and in magnitude the component displacement due to the initial velocity in t units of time. Let QR represent the component displacement due to the acceleration in t units of time. Then R is the position of the point after that time. Now $PQ = Vt$ and $QR = \frac{1}{2}at^2$. Eliminating t, we find

$$PQ^2 = (2V^2/a)QR.$$

This relation must hold for all values of t and therefore for all positions of the moving point. But we know, from the geometry of the parabola, that, if S is the focus

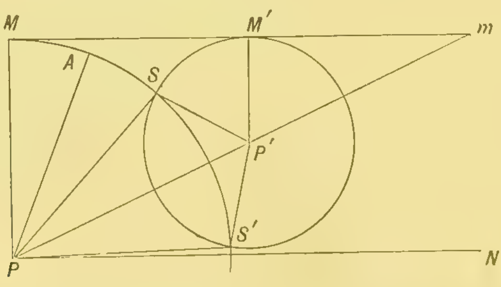
of a parabola which touches QP in P and whose axis is parallel to QR,

$$PQ^2 = 4SP \cdot QR.$$

Hence the path of the moving point is a parabola which touches PQ in P, has an axis parallel to QR, and has a focus distant $V^2/2a$ from P. To find the directrix of the parabola, we know that it must be perpendicular to QR and at a distance from P equal to PS. Hence from P draw PM parallel to QR and make it equal to $V^2/2a$. Then from M draw MM' perpendicular to PM. MM' is the directrix. To find the focus S we know that PS and PM must be equally inclined to PQ. Hence from P draw PS, making the angle SPQ equal to MPQ, and make PS equal to $V^2/2a$. S is the focus of the parabola. The directrix and focus being thus known, the parabolic path is known.

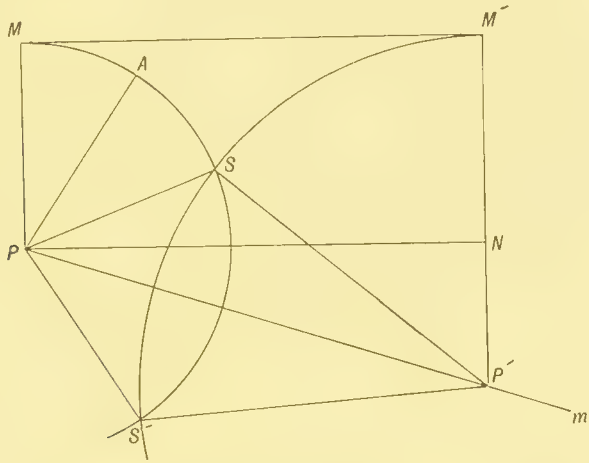
153. The acceleration and the magnitude of the initial velocity being given, PM will be constant. The length of PS will also be constant, but its direction will vary with the direction of the initial velocity. Hence the different positions occupied by S, for different directions of the initial velocity, lie on a circle whose centre is P and radius PS.

154. We may apply the geometrical method to deter-



mine the displacement in a given direction with given acceleration and initial velocity. Let P be the initial position of the point, PA the direction of the initial

velocity, MM' the directrix of the path. Then the focus of the path must lie on the circle MAS. If then the angle APS is made equal to the angle MPA, S is the



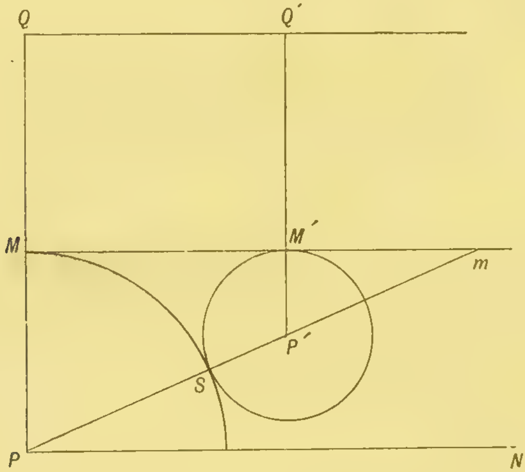
focus. To find at what point the path cuts the given direction Pm, it is necessary to find the point P' in Pm, whose distance from S is equal to its distance from MM', *i.e.*, to find in Pm the centre of a circle which will pass through S and touch MM'. Let P' be this point and $M'SS'$ the circle. Then $P'M' = P'S$. Hence P' is a point on the parabola whose focus is S, and therefore PP' is the displacement of the point in the direction Pm.

As the circles meet in general in two points, we have $MP = PS'$, and $M'P' = P'S'$. Hence P and P' are also points on the parabola whose focus is S'. Hence there are two paths by which, with a given acceleration and an initial velocity of given magnitude, a given distance in a given direction may be attained. The direction of the initial velocity which gives the second path is that of the line bisecting the angle $S'PM$.

As the angle MPA increases, S and S' approach one another and PP' increases in length. When $MPA = APm$, S and S' coincide, the circles merely touch,

and $PP' = PS + SP'$. This is the greatest distance the moving point can attain in the given direction with an initial velocity of given magnitude; and it can be attained obviously by one path only.

The locus of P', when PP' is the greatest distance which can be attained in different directions with an initial velocity of given magnitude, is the curve inside which all points can be reached by the moving point with an initial velocity of the given magnitude, outside which no points can be reached. It is evidently a para-



bola whose focus is P and vertex M. For if PM be produced to Q, so that $PM = MQ$, and if QQ' be drawn parallel to MM', and $P'M'$ produced to meet it in Q', we have $P'P = P'Q'$. Hence P' is a point on a parabola whose focus is P and directrix QQ', and whose vertex consequently is M.

155. *Examples.*

(1) A body is projected with an initial velocity of 30 ft. per sec. inclined 60° to the horizon. Find the velocity after 20 sec.

* In the solution of these problems the resistance of the air is not taken into account. When the value of g is not specified it is to be taken as 32·2 ft.-sec. units or 981 cm.-sec. units.

Ans. 618·2 ft. per sec. inclined 148° 36′·6 to the direction of the initial velocity.

(2) Find the direction and magnitude of the velocity of projection in order that the projectile may reach its maximum height at a point whose horizontal and vertical distances from the starting point are b and h respectively.

Ans. Direction inclined $\tan^{-1}(2h/b)$ to the horizon, magnitude $[(4h^2+b^2)g/2h]^{\frac{1}{2}}$.

(3) A particle is projected horizontally with a speed of 32·2 ft. per sec. from a point 128·8 feet from the ground. Find the direction of its motion when it has fallen half way to the ground.

Ans. Inclination to the vertical $=\tan^{-1}\frac{1}{2}$.

(4) A stone is let fall in a railway carriage travelling at the rate of 30 mls. per hour. Find its displacement relative to the road at the end of 0·1 sec.

Ans. 4·4028...ft. inclined 2° 3′·4 to the horizon.

(5) A stone is thrown into the air at an angle of 45° to the horizontal plane with a speed of 50 ft. per sec. Find the magnitude of the displacement at the instant at which the stone's velocity is horizontal.

Ans. 43·4... ft.

(6) A gun is fired horizontally at a height of 144·9 ft. above the surface of a lake and gives the ball an initial speed of 1,000 ft. per sec. Find (a) after what time, and (b) at what horizontal distance, the ball will strike the lake.

Ans. (a) 3 sec.; (b) 3,000 ft.

(7) A stone thrown at an elevation of 19° from the top of a tower falls in 5 sec. at a distance of 100 ft. from the base. Find (a) the height of the tower, and (b) the speed of projection.

Ans. (a) 368·06... ft.; (b) 21·15... ft. per sec.

(8) The elevation of a projectile is that of maximum range on a horizontal plane. Show that the time which elapses before it reaches a point in its path whose horizontal and vertical distances from its starting point are h and k respectively is $\left(\frac{2}{g}(h-k)\right)^{\frac{1}{2}}$.

(9) Three particles are projected at the same instant from the same point in different directions. Show that the area of the triangle of which they form the angular points varies as the square of the time, and that the plane passing through them remains parallel to itself.

(10) The velocities of a projectile at any two points of its path being given, find the difference of their altitudes above a horizontal plane.

Ans. $(V^2 - V'^2)/2g$, where V, V' are the magnitudes of the given velocities.

(11) A ball is projected with velocity of 100 ft. per sec. inclined 75° to the horizon. Find (*a*) the range on a horizontal plane; (*b*) the range on a plane inclined 30° to the horizon; and (*c*) what other directions of the initial velocity would give the same respective ranges.

Ans. (*a*) 155·2 ft.; (*b*) $(\sqrt{3}-1)207 \cdot 0\ldots$ ft.; (*c*) inclinations 15° and 45° respectively.

(12) At what elevation must a body be projected with a speed of 310·8 ft. per sec. that it may reach a balloon 500 ft. from the earth's surface and at a distance of 1,000 ft. from the point of projection?

Ans. Either 39° 17′·7 or 80° 42′·3.

(13) On a small planet a stone projected with a speed of 50 ft. per sec. is found to have a maximum range on a horizontal plane of 400 ft. Find the acceleration of falling bodies at the surface of that planet.

Ans. 6·25 ft.-sec. units.

(14) Show that with a given initial speed the greatest range on a horizontal plane is just half as great as the greatest range down an incline of 30°.

(15) The greatest range on a horizontal plane of a projectile with a given initial speed being 500 metres, show that the greatest range on a plane inclined 60° to the horizontal is $2-\sqrt{3}$ kilometres.

(16) AB being the range of a projectile on a horizontal plane.

show that if t be the time from A to any point P of the trajectory (*i.e.*, the path), and t' the time from P to B, the height of P above AB is $\frac{1}{2}gtt'$.

(17) A particle projected at a given elevation with an initial speed V reaches the top of a tower h ft. high and $2h$ ft. from the point of projection in t seconds. Find (a) the initial speed of another particle which, being projected at the same elevation from a point distant $4h$ ft. from the tower, will also reach its summit, and (b) the time it will require.

Ans. (a) $\sqrt{2g}\,Vt/(h+gt^2)^{\frac{1}{2}}$; (b) $[2(h+gt^2)/g]^{\frac{1}{2}}$.

(18) Two stones thrown at the same instant from points 20 yds. apart, with initial velocities inclined 60° and 30° respectively to the horizon, strike a flag-pole at the same point at the same instant. Show that their initial speeds are as $1 : \sqrt{3}$; and that the distance of the pole from the nearer point of projection is 10 yds.

(19) If a particle, projected with a speed u, strike at right angles a vertical wall whose distance from the point of projection is $u^2\cos\phi/2g$, prove that the angle of projection may be $\pi/4+\phi/2$ or $\pi/4-\phi/2$, and that the distance between the points at which it will strike the wall if projected at these elevations successively is $u^2\sin\phi/2g$.

(20) Show that if two particles meet, which have been projected with the same initial speed, in the same vertical plane, at the same instant, from two given points, the sum of their elevations must be constant.

(21) A particle is projected from a platform with a velocity V inclined α to the horizontal. On the platform is a telescope fixed at the elevation β. The platform moves horizontally in the plane of the particle's motion, so as to keep the particle in the centre of the field of view of the telescope. Show that the initial speed of the platform must be $V\sin(\beta-\alpha)/\sin\beta$, and its rate of change of speed $g\cot\beta$.

(22) In the parabola described by a projectile, its speed at any point is that which it would have had, had it fallen to that point from the directrix ; and the horizontal component of its velocity at any point is that which it would have had, had it fallen from rest through a distance equal to one-fourth of the *latus rectum*.

(23) The speed of a projectile at any point of its path is equal to that which it would have acquired had it fallen from rest through a distance equal to one-fourth of the focal chord, parallel to the direction of motion at the given point.

(24) If l is the length of a focal chord of the path of a projectile, show that the time of flight from one of its extremities to the other is $(2lg^{-1})^{\frac{1}{2}}$.

(25) If any number of bodies be projected from the same point in different directions and with equal speeds, prove that the foci of the parabolas they will describe will lie on the surface of a sphere.

(26) Particles are projected from the same point in horizontal directions and with different speeds. Show that the extremities of the *latera recta* of their paths will lie on a cone whose axis is vertical and whose vertical angle is $2\tan^{-1}2$.

(27) Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.

(28) Show that if a ball is projected from a point on an inclined plane in such a direction that its range on the plane is a maximum, the direction of its motion at the moment of striking the plane is perpendicular to the direction of projection.

(29) A sphere (radius $= r$) rests on a horizontal plane. Find at what distance from its point of contact with the plane a particle must be projected, with the speed which it would have gained in falling through a distance equal to the diameter of the sphere, in order that the focus of its path may be the centre of the sphere.

Ans. $\sqrt{h^2 - r^2}$.

156. *Case III.—Central Acceleration*, the acceleration directed towards a fixed point or centre. (See 138.)

If a point move under a central acceleration the moment of its velocity about the centre will be constant. —The velocity of the moving point at any instant is the resultant of the velocity at a former instant, and of the integral acceleration during the intervening time. During an indefinitely short time, the position of the moving point does not appreciably change. If therefore we draw, through any position occupied by the

point in such time, lines representing the initial and final velocities and the integral acceleration, the products of these lines into the perpendiculars on them from the fixed centre will (103) represent their respective moments about this centre. By 105 the moment of the final velocity about this centre will be equal to the algebraic sum of the moments of the initial velocity and the integral acceleration. But the moment of the acceleration about a point towards which it is directed is zero. Hence the moment of the velocity of the moving point about the centre is constant.

It is clear also that the converse proposition holds, that if the moment of the velocity of a moving point about any fixed point be constant, its acceleration must be directed towards the fixed point.

It follows from 132 that, if ω be the angular velocity of the moving point about, and r its distance from, the centre of acceleration, ωr^2, and therefore $\frac{1}{2}\omega r^2$, are constant. Hence (133) the areal velocity of the moving point, or the area described per unit of time by the radius vector from the centre of acceleration, is constant.

157. *Examples.*

(1) Various particles, whose accelerations are all directed to one centre C, are projected from a given point A with equal speeds but in different directions. Show that the areas described in a given time by lines drawn from C to the particles will be proportional to the sines of the inclinations of their initial velocities to the line AC. [The areal velocities are proportional to the moments of the linear velocities, and the perpendiculars on the directions of motion are proportional to the sines of the inclinations.]

(2) A point moves in an elliptic path with an acceleration directed to one of the foci. Show that its velocity varies inversely as the square root of its distance from that focus, and directly as the square root of its distance from the other, and has maximum and minimum values when the point is nearest to and farthest from the centre of acceleration respectively. [Note that the product of the perpen-

diculars from the foci on a tangent is equal to the square of the semi-axis minor.]

(3) A point moves in a parabola under an acceleration directed towards the vertex. Show that the time required to move from any point to the vertex will be found to vary as the cube of the distance of the point from the axis. [If P is a point on a parabola whose vertex is A, and if PM is a perpendicular on the axis of the parabola, the area APM is proportional to the product of AM into MP.]

(4) If an ellipse be described by a point under an acceleration directed towards its centre, the velocity of the point will vary directly as the diameter conjugate to that which passes through the point.

(5) A point moves in an ellipse $ABA'B'$ (major axis, $ASS'A'$; minor axis, BB'; foci, S and S') with an acceleration directed towards S. Show that the ratio of the times of describing AB and BA' is $(\pi - 2e)/(\pi + 2e)$, where e is the excentricity of the ellipse.

(6) A point moves in a circle and is observed to occupy, in passing from a fixed point in the circle to any other point, a time proportional to the sum of the lengths of the arc described and of the perpendicular from one extremity on the diameter through the other. Show that the acceleration of the moving point is directed towards a fixed point.

(7) Find the angular velocity of a point moving with a central acceleration, about the centre, in terms of the length of the radius vector (r) and the areal velocity (h).

Ans. $2h/r^2$.

158. We shall discuss two important cases of central acceleration, viz., that of planetary motion and that of harmonic motion.

I. *Planetary motion*, the acceleration being inversely proportional to the square of the distance of the moving point from the centre of acceleration. This case is of interest, because it is that of the motion of planets about the sun and of satellites about their primaries.

(*a*) *The motion rectilinear*, the velocity being in the same straight line as the acceleration at any instant (140). Let s be the distance of the moving point from the centre of acceleration at any instant. Then if a be the acceleration at that distance, and k a constant, $a=-k/s^2$, the negative sign being used because the acceleration is towards the point from which the distance s is measured. If v be the speed at the instant under consideration, and v' the speed after an indefinitely short time τ,

$$a=(v'-v)/\tau=-k/s^2.$$

If s' is the distance after the time τ, $(s'-s)/\tau$ is the mean speed during τ; and as τ is indefinitely short, we may consider it equal either to v or to v'. Hence

$$v+v'=2(s'-s)/\tau.$$

Hence also $$(v+v')\frac{v'-v}{\tau}=-2\frac{k}{s^2}\cdot\frac{s'-s}{\tau}.$$

As τ is indefinitely small we may consider s^2 equal to ss'.

Hence $$v'^2-v^2=-2k\left(\frac{1}{s}-\frac{1}{s'}\right)=2k\left(\frac{1}{s'}-\frac{1}{s}\right).$$

Let V be the velocity of the point when at a distance S. Then the space between the positions, whose distances from the centre are s and S, may be divided into an indefinitely great number of parts by points whose distances from the centre are s_1, s_2, etc., s_n. In that case, if v_1, v_2, etc., v_n, are the velocities of the moving point when it is at the above distances respectively, we have

$$V^2-v^2_n=2k\left(\frac{1}{S}-\frac{1}{s_n}\right),\quad v^2_n-v^2_{n-1}=2k\left(\frac{1}{s_n}-\frac{1}{s_{n-1}}\right),$$

etc.,

$$v_1^2-v^2=2k\left(\frac{1}{s_1}-\frac{1}{s}\right).$$

Hence by addition we obtain : $V^2-v^2=2k\left(\frac{1}{S}-\frac{1}{s}\right)$.

Therefore $V^2-\frac{2k}{S}=v^2-\frac{2k}{s}$, or $v^2-\frac{2k}{s}=A$ (a constant).

If s_0 is the distance from the centre of acceleration at which the velocity becomes zero (the distance of the starting point, if the moving point start from rest), we have $0 = \frac{2k}{s_0} + A$, and $A = -\frac{2k}{s_0}$. Hence $v^2 = 2k\left(\frac{1}{s} - \frac{1}{s_0}\right)$; and the speed at any given distance from the centre of acceleration is thus expressed in terms of that distance.

159. We may apply the above to the case of the falling of bodies to the earth from great distances. For this purpose we must determine the value of k in this case. Now the acceleration of a falling body at the earth's surface, *i.e.*, at a distance equal to the earth's radius (R) from the centre of the earth, is g; and by Newton's law of gravitation the acceleration of a falling body is inversely proportional to the square of its distance from the earth's centre. Hence at a distance s we have

$$a = -g\frac{R^2}{s^2};$$

and therefore in this case $k = gR^2$. Hence, if v is the velocity of a falling body at a distance s from the earth's centre, its velocity at a distance s_0 having been zero,

$$v^2 = 2gR^2\left(\frac{1}{s} - \frac{1}{s_0}\right).$$

At the earth's surface therefore its velocity will be such that

$$v^2 = 2gR^2\left(\frac{1}{R} - \frac{1}{s_0}\right) = 2gR\left(1 - \frac{R}{s_0}\right).$$

If the point from which the body has fallen be a short distance h from the earth's surface, $s_0 = R + h$, and

$$v^2 = 2gR\left(1 - \frac{R}{R+h}\right) = 2gR\left(1 - \left(1 - \frac{h}{R} + \frac{h^2}{R^2} - \text{etc.}\right)\right).$$

If now h be sufficiently small $(h/R)^2$ and higher powers may be neglected. Hence we have $v^2 = 2gh$, the result obtained in 65.

If a body fall to the earth's surface from a very great (practically infinite) distance, we have $1/s_0 = 0$, and hence $v^2 = 2gR$.

160. *Examples.*

(1) The acceleration (expressed in ft.-sec. units) of a moving point towards a centre is four times the square of the reciprocal of its distance from the centre. If it start from rest at a distance of 6 ft., find its speed at a distance of 1 ft.

Ans. 2·58... ft. per sec.

(2) A body falls to the earth from a point 1,000 mls. above its surface. Find its speed on reaching the surface (neglecting resistance of air and taking the earth's radius to be 4,000 mls.).

Ans. 3·12... mls. per sec.

(3) With the data of the last problem find the body's distance from the earth's surface when its speed is 2 mls. per sec.

Ans. 535·2... mls.

(4) With what speed must a body be projected vertically at the earth's surface that it may never return? (Assume the earth to have no atmosphere and not to be rotating.)

Ans. 6·98... mls. per sec.

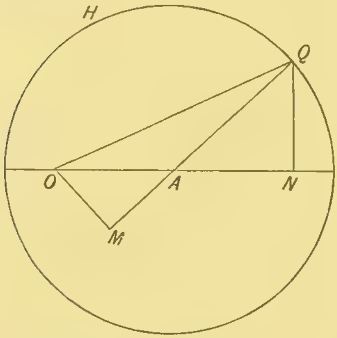
(5) At what point on a line joining the centres of the earth and moon will a body have no acceleration? (Acceleration of falling bodies at the moon's surface due to moon's attraction = 5·5 ft.-sec. units; radius of moon = 1,080 mls.; distance between centres of earth and moon = 240,000 mls.)

Ans. At a point about 215,900 mls. from the earth's centre.

161. (*b*) *The motion curvilinear*, the velocity at any instant being inclined to the acceleration.

As ωr^2 is constant (156), the angular velocity of the moving point P about the centre of acceleration is proportional to $1/r^2$, and therefore to its linear acceleration. Now the angular velocity of P is also the angular velocity of the direction of the acceleration, and is therefore (112) equal to the angular velocity of the tangent at the corresponding point Q of the hodograph. And the linear acceleration of P is equal to the linear velocity of the

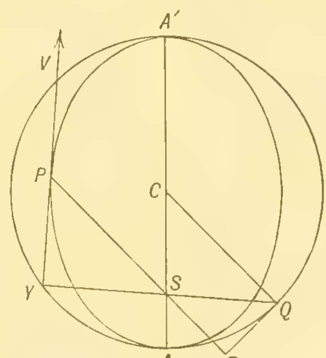
point Q in the hodograph. If therefore s be the length of the small arc between two points of the hodograph, and ϕ the angle between the tangents at these points, ϕ/s is constant. Now the acceleration of P is in the same plane as its velocity at any instant and the centre of acceleration, and therefore its path also, is in that plane. Hence the hodograph is a plane curve of constant curvature, *i.e.* (40), a circle. Let H be the circular hodograph, O its pole (which may be either inside or outside or upon the circumference), A its centre, and Q the point in it corresponding to the position P of the moving point in its path. Through O draw OM perpendicular to QA or QA produced, and through Q draw QN perpendicular to OA or OA produced.



Since the tangent at Q is in the direction of the acceleration of P, and therefore in that of the radius vector, OM is the component of the velocity of P, in the direction of the radius vector, and is therefore clearly equal to the rate of change of the length of the radius vector. Also QN is the component, perpendicular to the fixed line OA, of the velocity of P. Hence the ratio of OM to QN is the ratio of a small increment of the radius vector to the simultaneous increment of the distance of the point P from a fixed line in the plane of motion. Now the triangles OAM and QAN are similar, and the ratio of OM to QN is therefore equal to the ratio of OA to AQ, and consequently is constant. P's path is therefore such that if r and r' are initial and final values of the radius vector in a short time, and if d and d' are corresponding values of the distance of P from a certain fixed line in the plane of motion, $(r'-r)/(d'-d)=k$ (a constant). Take another fixed line parallel to the given fixed line, and so placed in the plane of motion that, if P's distance from it is δ, when the radius vector is r, we

may have $r/\delta = k$. Also, when the radius vector is r', let δ' be the distance of the point from this line. Then $d'-d=\delta'-\delta$. Hence $r'-r=k(\delta'-\delta)$. Now $r=k\delta$. Therefore $r'=k\delta'$. Hence the ratio of the distance of the moving point from a fixed point to its distance from a fixed line has a constant value, or, in other words, the path must be a conic section. If $k<1$ (and therefore the point O inside the circle), the path is an ellipse. If $k=1$ (the point O on the circumference) it is a parabola. If $k>1$ (O outside the circle) it is an hyperbola.*

162. The astronomical problem is the converse of the above. Kepler generalized from many series of observations (1) that the path of each planet is an ellipse, one of whose foci is occupied by the sun; and (2) that the radius vector of each planet, from the sun, describes equal areas in equal times. These are two of what are known as Kepler's laws. In astronomy, therefore, we have to determine the direction and magnitude of the acceleration of a point whose path is an ellipse and whose radius vector from one focus describes equal areas in equal times. Let P be the position of the planet at any instant, V its velocity, APA' its elliptic path, AA' the axis major of the path, S the focus occupied by the sun, and SY a perpendicular from S on the tangent at P. The locus of Y is a circle on AA' as diameter. Draw this circle and let YS meet it in Q.



By the second of Kepler's laws, $V.SY$ is constant (132-3), equal to h, say, and by a property of the circle

* This proof is due to Prof. Tait. See Encyclopædia Britannica, 9th ed., art. *Mechanics.*

$SY.SQ$ is also constant, and being equal to $AS.SA'$ is by a property of the ellipse equal to b^2, b being the semi-axis minor. Hence SQ is equal to b^2V/h. And it is at right angles to the direction of V. Hence the locus of Q, the circle AYA', turned through a right angle about S so that SQ may become codirectional with V, is the hodograph of P's motion on the scale of h/b^2 to unit length. By a property of the ellipse, CQ is parallel to PS. Hence the tangent QE at Q, whose direction is that of Q's velocity, is perpendicular to PS, and, if the circle be turned through a right angle, will be codirectional with PS. But (113) the direction of the velocity of Q is that of the acceleration of P. Hence P's acceleration is towards S.

Also the magnitude of P's acceleration is equal to h/b^2 times that of Q's velocity. And a being the semi-axis major, Q's velocity is equal to a times the angular velocity of Q about C, *i.e.*, since CQ and PS are parallel, of P about S. And the areal velocity of P, and therefore the moment of its velocity, about S being constant, its angular velocity about S is equal to h/SP^2. Hence the acceleration of P is equal to $\frac{h^2a}{b^2}\cdot\frac{1}{SP^2}$, and is therefore inversely proportional to its distance from S.*

The velocity at any point of the orbit may readily be expressed in terms of the radius vector SP or r. The moment of the velocity being constant, we have $V=h/p$, if p is the perpendicular from S on the velocity line. Hence if we indicate the constant h^2a/b^2 by the symbol μ, we have $V^2=b^2\mu/ap^2$. If p', r' are the perpendicular on the line of V and the radius vector to P, both drawn from the other focus, we have by the properties of the ellipse

$$r+r'=2a,\ pp'=b^2,\ p'/p=r'/r.$$

* This proof is also due to Prof. Tait. See his "Properties of Matter," § 146.

Hence $$V^2=\frac{\mu p'}{ap}=\frac{\mu r'}{ar}=\frac{\mu}{a}\cdot\frac{2a-r}{r}=\mu\left(\frac{2}{r}-\frac{1}{a}\right).$$

The periodic time T in the orbit will be equal to its area divided by the areal velocity. Hence

$$T=\frac{\pi ab}{h/2}=\frac{2\pi ab}{\sqrt{\mu b^2/a}}=2\pi a^{\frac{3}{2}}/\mu^{\frac{1}{2}}.$$

163. *II. Harmonic motion,* the magnitude of the central acceleration being directly proportional to the distance of the moving point from the centre of acceleration.—This case is of interest because it is that of the motion of elastic bodies after compression or distortion. It includes therefore the motion of air and of the luminiferous ether in the transmission of sound and light respectively.

(a) *The motion rectilinear*, the velocity at any instant being in the same straight line as the acceleration (140) —*simple harmonic motion.*

Let a be the acceleration of the moving point when at a distance s from the centre of acceleration. Then, k being a constant, $a=-ks$, the negative sign being used, as in 158. Let the point move to a position at a distance s' from the centre. Then, since the acceleration increases uniformly with the distance, its average value per unit distance during the above displacement must be half the sum of its initial and final values, *i.e.*, $-k(s+s')/2$. The change of velocity during the displacement is the same as if the point had had an acceleration of this amount during the whole displacement. Hence, if v, v' are the velocities of the moving point at the distances s, s' respectively (140, 65),

$$v'^2-v^2=2\left(-\frac{k}{2}(s+s')\right)(s'-s)$$
$$=k(s^2-s'^2).$$

As the point moves away from the centre its velocity diminishes. Let s_0 be the distance at which it becomes

zero. Then at any other point distant s its velocity v is such that

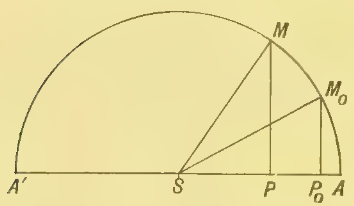
$$v^2 = k(s_0^2 - s^2).$$

When the point reaches the centre of acceleration, $s=0$, and $v^2 = ks_0^2$. Hence its speed on passing through the centre is $\sqrt{k}\,.\,s_0$. At any point distant $-s$ from the centre its speed is such that

$$v^2 = k(s_0^2 - s^2)$$

and is therefore the same as at a point distant $+s$. At a point distant $-s_0$ its speed is zero. Hence the moving point starting from a distance s_0, with zero speed, moves with increasing speed to the centre of acceleration where its speed is $\sqrt{k}\,.\,s_0$; thence with decreasing speed to a distance $-s_0$; and thence back to the starting point, undergoing the same changes of speed in the reverse order; and so on, its whole motion consisting of a series of such oscillations.

Let S be the centre of acceleration, SA the line of motion. From S as centre with a radius equal to s_0 describe a circle. From P, whose distance from S is s, draw PM perpendicular to SA and meeting the circle in M. If now the point M move with a uniform speed $\sqrt{k}\,.\,s_0$ in the circle, P, the foot of the perpendicular from M on SA, will move in SA with a speed which is the component of M's velocity in the line SA and is therefore



$$\sqrt{k}\,.\,s_0 \cos SMP = \sqrt{k}\,.\,s_0 \frac{PM}{SM} = \sqrt{k}\sqrt{s_0^2 - s^2}.$$

If then P's velocity is v,

$$v^2 = k(s_0^2 - s^2).$$

Hence P's velocity, and consequently also its acceleration, at any given distance from S, are the same as the velocity and acceleration respectively of the moving point under consideration when at the same distance from its centre

of acceleration. Hence the motion of a point moving in a straight line with an acceleration directly proportional to its distance from a centre of acceleration in that line is the resolved part in the direction of that line of the motion of a point moving with uniform speed $\sqrt{k} \cdot s_0$ in a circle whose centre is the centre of acceleration and whose radius is s_0.

The time required by the moving point to make a complete oscillation from A to A' and back to A being that required by M to move once round the auxiliary circle is clearly

$$\frac{2\pi s_0}{\sqrt{k} \cdot s_0} = \frac{2\pi}{\sqrt{k}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}},$$

since the magnitude of k is the ratio of the acceleration of the point to its displacement, in any position. The time of a complete oscillation depends therefore only upon the value of k, the constant ratio of the acceleration of the moving point to its displacement from the centre of acceleration. It is independent of the extent of the oscillation. For this reason such oscillations are said to be isochronous.

The time required by the moving point to move from a position P_0 to P is, if M_0 is the intersection with the circle of a line drawn from P_0 perpendicular to SA,

$$\frac{2\pi}{\sqrt{k}} \cdot \frac{\text{angle } M_0SM}{2\pi} = \frac{1}{\sqrt{k}} \cdot \text{angle } M_0SM,$$

the angle being measured in radians.

The oscillation of a point moving in a straight line about a fixed point in the line towards which its acceleration is directed, the acceleration being directly proportional to the distance between the points, is called Simple Harmonic Motion.*

* Simple Harmonic Motion is thus not only the simplest form of the motion of bodies after release from strain, but is also the apparent motion of bodies moving in circular orbits when observed from a distant point in the plane of the orbit, as, *e.g.*, approximately in the case of the motion of Jupiter's Satellites.

164. It will be obvious that the above results apply also to the case of a point moving in a curved path, provided its rate of change of speed is proportional to its distance (measured along the path) from a fixed point in the path, and is positive or negative, according as it is moving towards or from the fixed point.

165. The distance of the centre of acceleration or the mean position of the moving point P from its extreme position, SA in the figure of 163, is called the *Amplitude* of the simple harmonic motion. The interval of time between two successive passages of the moving point through the same position in the same direction is called the *Period.* The fraction of the period intervening between the instant of the point's occupying its extreme position A in the positive direction, and that at which it occupies any given position is called the *Phase.* The phase is frequently described by reference to the auxiliary circle. In that case it is defined either as the angle ASM (P, 163, being the given position of the moving point) or as the ratio of the angle ASM to the whole angle (2π radians) through which SM moves in the period. The *Epoch* is the phase at zero of time, and is specified in the same way as the phase. The epoch determines the position of the point at zero of time, the phase its position after any given interval. The epoch has a definite value in any given case of simple harmonic motion, the phase varies with the time.

166. *Examples.*

(1) A point whose motion is simple harmonic has velocities 20 and 25 ft. per sec. at distances 10 and 8 ft. respectively from its centre of acceleration. Find (a) its period, and (b) its acceleration at unit distance from the centre.

Ans. (a) $\frac{4\pi}{5}$ sec.; (b) 6·25 ft.-sec. units.

(2) The period of a simple harmonic motion is 20 sec. and the

maximum velocity of the moving point is 10 ft. per sec. Find its velocity at a distance of $60/\pi$ ft. from the mean position.

Ans. 8 ft. per sec.

(3) A point moves from rest towards a fixed point 10 metres distant, its acceleration being everywhere 4 times its distance from the fixed point. At what distance will it have a velocity of 12 metres per sec.?

Ans. 8 metres.

(4) Find the mean speed of a point executing a simple harmonic motion, during the time occupied in moving from one to the other extremity of its range, its maximum speed being 5 ft. per sec.

Ans. $10/\pi$ ft. per sec.

(5) If T be the period and a the amplitude of a simple harmonic motion, and if v be the velocity and s the distance from the centre, of the moving point at a given instant, show that

$$a = \left(\frac{T^2 v^2}{4\pi^2} + s^2\right)^{\frac{1}{2}}.$$

(6) A point oscillates about a centre, its acceleration being proportional to its distance from the centre. Show that the ratio of its maximum velocity to the square root of the excess of the square of its maximum velocity over the square of the velocity which it has when at a given displacement from the centre, is equal to the ratio of its maximum displacement to the given displacement.

(7) A point has a simple harmonic motion whose period is 4 min. 12 sec. Find the time during which its phase changes from $\frac{1}{12}$ to $\frac{1}{6}$ of a period.

Ans. 21 sec.

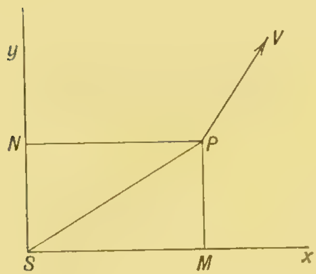
(8) A moving point has a velocity of 1 ft. per sec. when at a distance of $\sqrt{3}$ ft. from a fixed point in its line of motion towards which its acceleration and motion are directed, its acceleration being everywhere numerically equal to its distance from that point. After what time will it be at a distance of 1 ft.?

Ans. $\pi/6$ sec.

(9) Show that a point having a simple harmonic motion requires

$\frac{1}{6}$ of its period to move from a position in which its displacement is a maximum to one in which its displacement is one-half the amplitude.

167. (*b*) *Curvilinear motion*, the velocity of the moving point at any instant being inclined to the acceleration—*compound harmonic motion*.



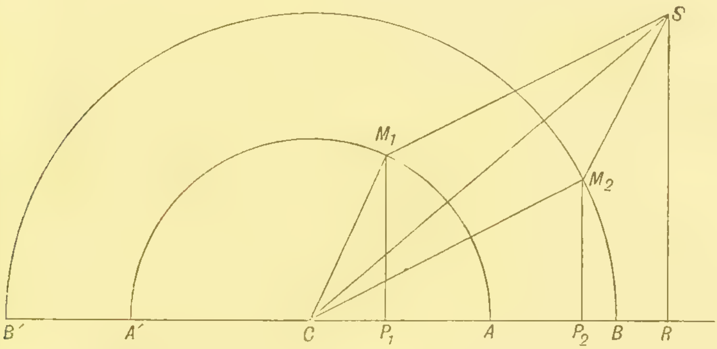
Let S be the centre of acceleration, P the position of the moving point at any instant, and V its velocity at that instant. In the plane of V and SP take two fixed rectangular axes Sx, Sy. From P draw PM, PN perpendiculars on Sx and Sy respectively. Let the inclination of V to Sx be α. Then the moving point has in the direction of Sx a component velocity $V\cos\alpha$, and, if s is the distance of P from S, a component acceleration

$$-ks\cos PSM = -ks\frac{SM}{SP} = -k\,.\,SM.$$

Similarly in the direction of Sy, P has a component velocity $V\sin\alpha$ and a component acceleration $-k\,.\,SN$. Hence the motion of the moving point is the resultant of two component simple harmonic motions, the one in the direction Sx, the other in the direction Sy. We may therefore determine its motion by determining the laws of the composition of simple harmonic motions. We shall investigate these laws at greater length than is necessary for the mere solution of the above problem, as they are of great importance in the study of sound and light.

168. *Composition of Simple Harmonic Motions*.—A point has two or more component simple harmonic motions; it is required to determine its resultant motion.

(1) *Two Simple Harmonic Motions in the same line and with the same period.*—Let the point P, moving in the line BB', have two component simple harmonic



motions, of amplitudes CA and CB, and of the same period. Let CP_1 and CP_2 be the component displacements due to the respective simple harmonic motions at a given instant. Then the resultant displacement is (86, III.) $CP_1 + CP_2$. Draw the auxiliary circles, and let M_1, M_2 be the points in these circles corresponding to P_1, P_2. Complete the parallelogram M_1M_2, and from S draw SR perpendicular to BB'.

Since $M_1C = SM_2$ and angle $CM_1P_1 = M_2SR$, $CP_1 = P_2R$. Hence CR is equal to the resultant displacement, and R's motion is the resultant motion. Since the periods of the motions are the same, the angular velocities of CM_1 and CM_2 are the same. Hence the angle M_1CM_2 is constant, and therefore the length of the diagonal CS of the parallelogram M_1M_2, and its inclination to CM_1 or CM_2, are constant. S therefore moves with uniform speed in a circle. Hence R's motion is simple harmonic, and therefore the resultant of two simple harmonic motions in the same line and of the same period is also a simple harmonic motion.

As the inclination of CS to CM_1 is constant, the period of the resultant simple harmonic motion is the common

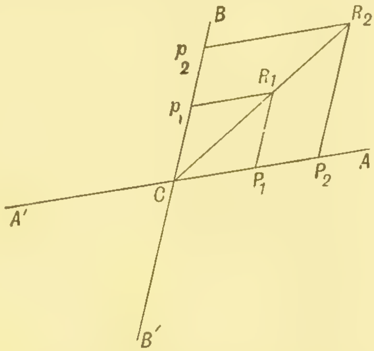
period of the components. Its amplitude is CS. Its phase is intermediate between the phases of the components. If the phases of the two components are the same, the amplitude of the resultant motion is the sum of those of the component motions. If the difference of phase is $\frac{1}{2}$ period $\left(\text{or } \frac{2n+1}{2} \text{ periods}\right)$, the amplitude of the resultant is the difference of those of the components.

169. As, by taking CM_1 and CM_2 of proper lengths, the angles M_2CP_2 and M_1CM_2 may be made what we please, while CS is kept constant, any given simple harmonic motion may be resolved into two components in the same line, having any desired difference of phase, and one of them having any desired epoch.

170. (2) *Three or more component Simple Harmonic Motions in the same line and of the same period* may be compounded two and two, the above process being applied in each case. The resultant motion will evidently be simple harmonic of the period common to the components.

171. (3) *Two component Simple Harmonic Motions in the same line but of different periods.*—If the periods are not the same, the angle M_1CM_2 (168), and consequently also CS, are variable. At the instants at which the phases of the component motions are the same or differ by n periods, CS has its maximum value, viz., $CM_1 + CM_2$. At the instants at which the phases differ by $\frac{2n+1}{2}$ periods, CS has its minimum value, viz., $CM_2 - CM_1$. The angular velocity of CS will also be variable. The direction of CS will oscillate back and forth about that of CM_2, their maximum inclination being $\sin^{-1}(CM_1/CM_2)$. The resultant motion is therefore not simple harmonic but a more complex motion.

172. (4) *Component Simple Harmonic Motions in different lines with the same period and phase.*—Let the point P have two component simple harmonic motions in the lines AA' and BB'. Let CP_1 and CP_2, and Cp_1 and Cp_2 be the component displacements of P at times t_1 and t_2, due to the respective component motions. Then, as periods and phases are the same, $CP_1/CP_2 = Cp_1/Cp_2$. Complete the parallelograms p_1P_1, p_2P_2. Then CR_1 and CR_2 are in the same straight line and $CP_1/CP_2 = CR_1/CR_2$; *i.e.*, the resultant motion is a simple harmonic motion in the line CR_2, and is of the same period and phase as the components. The amplitude is the diagonal of the parallelogram, whose adjacent sides represent the amplitudes of the components and are inclined at the inclination of the lines in which the simple harmonic motions occur.

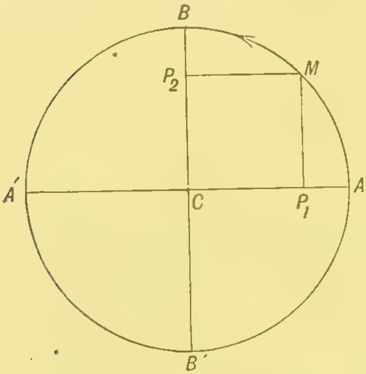


Hence a simple harmonic motion may be resolved in any two directions into two simple harmonic motions of the same period and phase as the given simple harmonic motion.

It follows that the projection of a simple harmonic motion on any straight line or on any plane is also a simple harmonic motion of the same period and phase as the projected simple harmonic motion.

If the component simple harmonic motions are more than two, they may be compounded two by two according to the above law, and it follows that any number of component simple harmonic motions, in any directions, and of the same period and phase, give as resultant a simple harmonic motion of the same period and phase in a determinate direction and of determinate amplitude.

173. (5) *Two Component Simple Harmonic Motions in different lines with the same period but with different phases.*—We have seen (163) that if a point move uniformly in a circle, the component of its motion in the direction of a diameter is a simple harmonic motion. Hence the uniform motion of a point in a circle may be resolved into two simple harmonic motions in directions at right angles to one another. These simple harmonic motions will clearly have the same periods and amplitudes. They will differ in phase however by one quarter of a period. For let AA', BB' be perpendicular diameters of the circle $ABA'B'$, in which the point M is moving counter-clockwise. Then the foot P_1 of the perpendicular MP_1 will be moving towards C, while P_2, the foot of the perpendicular MP_2, is moving towards B. When P_1 is at A (*i.e.*, has the phase zero), P_2 will be at C, and not until M has moved from A to B will P_2 have the phase zero.



It follows also that two component simple harmonic motions in perpendicular directions, of the same period, of equal amplitudes, and with phases differing by one quarter of a period, will give as resultant, uniform motion in a circle whose radius is the common amplitude of the components.

Now the orthogonal projection of a circle is an ellipse,* the centre of the circle projecting into the centre of the ellipse; the projection of uniform motion in a circle (a motion in which the areal velocity about the centre is constant) is motion in an ellipse with constant areal

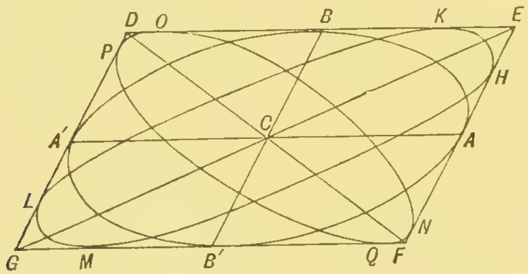
* If the reader is not familiar with the geometry of projection, he should read the chapter on this subject in Todhunter's "Conic Sections" or some similar work.

velocity about the centre: the projections of perpendicular diameters of a circle are conjugate diameters of the ellipse, whose inclination and relative length may be made what we please by a proper selection of the plane of projection: and the projection of a simple harmonic motion we have seen (172) to be a simple harmonic motion with unchanged period and phase. If, therefore, we project the circle $A'BAB'$, with its perpendicular diameters, on a plane so selected that the projections of the diameters have any desired inclination and relative length, the projections of the motions of P_1 and P_2 will be simple harmonic motions differing in phase by a quarter of a period; and their resultant motion, the projection of that of M, will be motion in the ellipse which is the projection of $A'BAB'$, the motion being such that the areal velocity of the moving point about the centre of the ellipse is constant. Hence, if a point have two component simple harmonic motions in any directions, of any amplitudes, of the same period, and with phases differing by a quarter of a period, the resultant motion will be motion in an ellipse, with conjugate diameters whose directions are the directions, and whose lengths are twice the amplitudes, of the component motions, and with constant areal velocity about the centre. Such a motion is called *elliptic harmonic motion.*

174. If now the two component simple harmonic motions differ in phase by any amount, each of them may (169) be resolved into two in its own direction, which differ in phase by a quarter of a period, and one of which has any desired epoch. Thus we have now two pairs of components, the components of each pair having the same phase, but differing in phase by a quarter of a period from those of the other pair. The components of each pair give as resultant a simple harmonic motion of determinate amplitude and direction, and of their common period and phase. Hence we obtain two simple harmonic motions of determinate amplitude and direction, equal in

period, and differing in phase by one quarter of a period, the resultant of which is determined by 173. Hence the resultant of two component simple harmonic motions of the same period, whatever may be their amplitudes, directions, or phases, is elliptic harmonic motion.

175. It will be obvious that all possible paths of a point having two such component simple harmonic motions, represented by AA' and BB', must touch each of the sides of a parallelogram $DEFG$, whose sides pass through



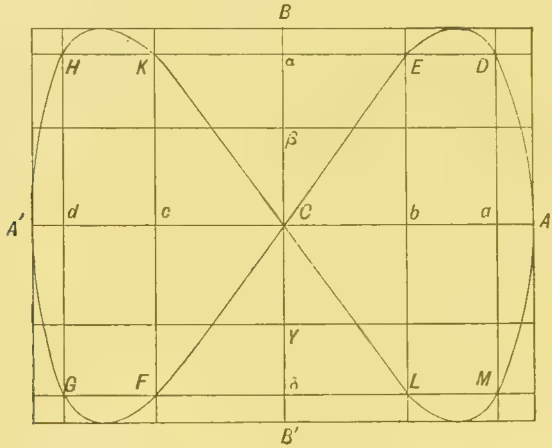
A, A', B, B', and are parallel to AA' and BB'. What the particular path will be, with amplitudes and directions given, will depend upon the difference of the phases of the components. If there is no difference of phase the path is the diagonal GE. If the phases differ by one quarter of a period (that of the simple harmonic motion in AA' being ahead), the point will move in the ellipse $ABA'B'$, and its motion will be counter-clockwise. If they differ by one-half period, the diagonal FD will be the path. If by three-quarters, the point will again move in the ellipse $ABA'B'$, but its motion will be clockwise. For differences of phase of intermediate value the paths will be ellipses in intermediate positions. Thus, for differences between 0 and $\frac{1}{4}$ or $\frac{3}{4}$ and 0, the paths will be such ellipses as $HKLM$, the motion being counter-clockwise or clockwise respectively; and for differences between $\frac{1}{4}$ and $\frac{1}{2}$ or $\frac{1}{2}$ and $\frac{3}{4}$ such ellipses as $NOPQ$ traversed counter-clockwise or clockwise respectively.

176. (6) *Three or more component Simple Harmonic Motions in different lines with the same period but with different phases.*—If there be more than two component simple harmonic motions of the same period, but in different lines, and of different amplitudes and phases, each of them may, as in 174, be resolved into two in its own direction, which differ in phase by a quarter of a period, and one of which has any desired epoch. We thus obtain two sets of component motions, all the members of each set having the same phase, but the members of each set differing in phase from those of the other set by a quarter of a period. The components of each of these sets give, when compounded (172), a simple harmonic motion in a determinate direction, of determinate amplitude, and with the common phase of its components. Hence we obtain two simple harmonic motions in known directions, of known amplitudes, and differing from one another in phase by one quarter of a period. The resultant motion is therefore determined by 173. Hence the resultant of any number of component simple harmonic motions of the same period, whatever their amplitudes, directions, or phases, is elliptic harmonic motion.

177. (7) *Component Simple Harmonic Motions differing in period.*—If a point have two or more component simple harmonic motions differing in period, the complete determination of the resultant motion is not possible by elementary mathematical methods. The path of the point may however always be found graphically by determining its positions at a series of instants and drawing a curve through them.

For example, let us find the path of a point P which has two component simple harmonic motions in lines at right angles to one another, with periods as $1:2$, the simple harmonic motion of longer period having zero epoch, and that of shorter period an epoch $3\pi/2$. Let AA', BB' be the given lines at right angles to one another,

CA and CB the given amplitudes of the simple harmonic motions in these lines. Let the simple harmonic motion in AA' be the one of longer period. The component displacement of P from C at zero of time in AA' is CA.



Since the epoch of the simple harmonic motion in BB' is $3\pi/2$, the component displacement of P from C in BB' at the same instant must be zero, and its component motion must be from C towards B. Hence the position of P at zero of time is A. To find other points in the path we may divide AA' and BB' into portions requiring equal times for their description. This may be done by describing semi-circles on AA' and BB' as diameters, dividing the semi-circles into a convenient number (say six) of equal arcs and dropping perpendiculars from the points of section on the respective diameters. Let $B\alpha$, $\alpha\beta$, βC, $C\gamma$, etc., be portions of BB', thus determined, requiring equal times for their description, and let Aa, ab, bC, etc., be similar portions of AA'. Through B, α, β, γ, δ, B' draw lines parallel to AA', and through A, a, b, c, d, A' draw lines parallel to BB'.

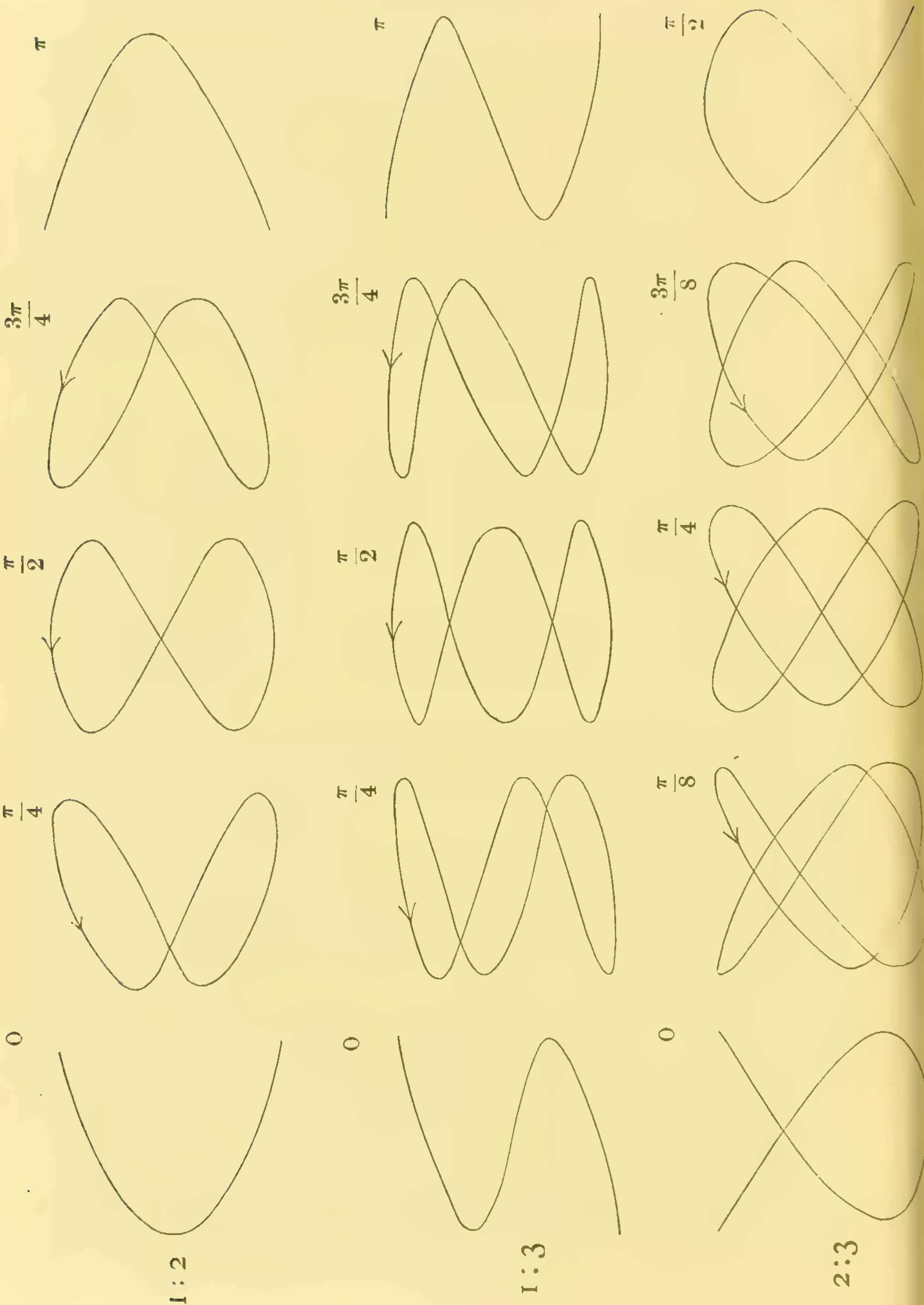
Since the simple harmonic motion in AA' is the one of longer period, a component displacement in AA' is accompanied by one in BB' of double the amount.

Hence, while P undergoes the displacement Aa in the line AA', it undergoes the displacement Ca in BB'. Hence P moves from A to D. Similarly the component displacements ab and $aB+Ba$ occur in the same time. Hence P moves from D to E. Similarly bC and aC, Cc and $C\delta$, cd and $\delta B'+B'\delta$, dA' and δC, $A'd$ and Ca, dc and $aB+Ba$, cC and aC, Cb and $C\delta$, ba and $\delta B'+B'\delta$, and aA and δC are pairs of displacements occurring in the same time. And hence the path passes through the following points in order, viz., A, D, E, C, F, G, A', H, K, C, L, M, A, and will be approximately represented by a smooth curve through these points.

The figures on next page represent a few paths of points having two component simple harmonic motions in lines at right angles to one another and differing in period and epoch. The ratio of the periods is indicated at the left of the row of figures to which it refers. The component simple harmonic motion of shorter period is horizontal. Its epoch is indicated in each case. The epoch of the vertical simple harmonic motion is zero.

178. If the periods of component simple harmonic motions are commensurable, at the end of a period which is their least common multiple the resultant displacement of the moving point from the mean position will be the same as at the beginning of the period; and the path will return into itself, forming a closed curve. If the periods be not commensurable, the path will not thus form a closed curve.

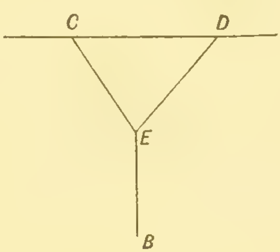
179. If the ratio of the periods of two component simple harmonic motions is very nearly a simple ratio, but not exactly, the path very nearly returns into itself; and it is clearly the same as if the periods were thus simply related, with the difference of epoch slowly changing. Hence the point will very nearly pass through



1 : 2
1 : 3
2 : 3
0
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{\pi}{8}$
$\frac{3\pi}{8}$

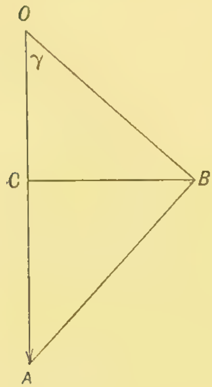
all the paths of a point with component simple harmonic motions having periods in the given ratio and of the given amplitudes and directions, with all differences of epoch. Thus, if the amplitudes and the directions are as represented in 175, the periods being very nearly equal, and if at a given instant the phases are the same, the point will first oscillate in a very elongated ellipse about *GE*. The ellipse will gradually open out through *HKLM* to *ABA'B'*, and passing through all such forms as *OPQN* will gradually come to oscillate in *DF*. It will then open out again and retrace nearly the same ellipses in the opposite direction, passing through *OPQN*, *ABA'B'*, and *HKLM* until it again oscillates in the line *GE*. Similarly, if the periods be very nearly as 1 : 2, or 1 : 3, or 2 : 3, the path of the moving point will gradually pass through the appropriate series of forms represented on p. 122.

180. Paths similar to those represented on page 122 may be traced out most simply by the aid of Blackburn's pendulum, which consists of a bob hung by a **Y**-shaped arrangement of wires *CDEB*, the ends *C* and *D* being attached at points in a horizontal line. Thus hung, the bob oscillates in a direction perpendicular to the plane *CDEB*, about the axis *CD*. In this plane it oscillates about *E*. Hence (187) *B* has two component simple harmonic motions at right angles to one another and of different periods. The difference of period may be made what we please by properly adjusting the lengths of the wires. If the bob be provided with a funnel containing sand or ink, it will leave a tracing of its path.



181. *Constrained Motion under given Accelerations.*—We take next certain cases of the motion of a point under conditions of constraint (see 138).

(1) *Motion on an Inclined Plane under Uniform Acceleration.*—Let a point having a uniform acceleration a, whose direction is OA, be constrained to remain in a plane whose inclination to OA is γ. From A draw AB perpendicular to the plane and meeting it in B. Then the angle AOB is γ. The effective component of the acceleration in the plane is $a \cos \gamma$ in the direction OB. For the component normal to the plane cannot affect motion in it. Hence the motion of the point will be rectilinear or parabolic according to the direction of the initial velocity, and will be determined by the equations of 140 and 142-150, $a \cos \gamma$ being the acceleration in the formulae of those articles instead of a. In the case in which the acceleration is that due to the weight of a body, OA is vertical and the given plane OB may have any inclination.



182. The speed gained by the point in moving on the given plane through the distance OB is equal to that which would be gained in moving, with the same initial speed, in the direction of the acceleration OA, through a distance which is the projection of OB on OA. To prove this, draw BC from B perpendicular to OA. Then, calling OB l, OC h, the initial speed V, and the speed at B v, we have

$$v^2 - V^2 = 2al \cos \gamma.$$

Had the point moved from O to C with the same initial speed its speed v' at C would have been such that

$$v'^2 - V^2 = 2ah = 2al \cos \gamma.$$

Hence $$v' = v.$$

183. The times required to produce these changes of speed are of course different. Thus, if t, t' are the times

required by the point to move from O to C and from O to B respectively, we have

$$v = V + at, \text{ and } v = V + at' \cos\gamma.$$

Hence $$t = t' \cos\gamma.$$

184. *Examples.*

(1) A point having a constant acceleration of 24 ft.-sec. units is constrained to move in a direction in which its speed changes in 1 min. from 10 to 250 yds. per sec. Find the inclination of its direction of motion to that of the given acceleration.

Ans. 60°.

(2) A heavy particle ($g = 32$) is projected* up an inclined plane whose inclination to the horizon is 30°. Find the distance traversed during a change of speed from 48 to 16 ft. per sec.

Ans. 64 ft.

(3) A railway carriage has, when 1 mile up an incline of 1 in 50 (*i.e.*, one having an inclination to the horizon of $\sin^{-1}\frac{1}{50}$), an upward velocity of 30 miles per hour. (a) In what time will it come to a standstill? (b) If it afterwards run back, with what speed will it reach the foot of the incline? (Take $g = 32$.)

Ans. (a) 1 min. 8·75 sec.; (b) 63·5 miles per hour.

(4) A body slides from rest down a smooth sloping roof and then falls to the ground. The length of the slope is 18 ft., its inclination to the horizon 30°, and the height of its lowest point from the ground 40 ft. Find the distance from the foot of the wall to the point where the body reaches the ground. (Take $g = 32$.)

Ans. $15\sqrt{3}$ ft.

(5) The times in which heavy particles slide from rest down inclined planes of equal height are proportional to their lengths. (The length of an inclined plane is the distance between its highest

* In these problems friction and other forms of resistance are not to be taken into account. Also, the motion on an inclined plane is always supposed to be in the direction of greatest slope, unless specially stated to be in some other direction.

and lowest points; its height is the distance between horizontal planes through these points.)

(6) If heavy particles slide down the sides of a right-angled triangle whose hypothenuse is vertical, they will acquire speeds proportional to the sides.

(7) The times required by heavy particles to descend in straight lines from the highest point in the circumference of a vertical circle to all other points in the circumference are the same.

For, if d is the diameter of the circle and θ the inclination to the vertical diameter of any chord through the highest point, the component acceleration in the direction of the chord is $g\cos\theta$, and the length of the chord is $d\cos\theta$. Hence, if t is the time in which a particle would fall from rest down this chord,

$$d\cos\theta = \tfrac{1}{2}gt^2\cos\theta \text{ and } t = \sqrt{2d/g}.$$

Thus t is independent of θ and is therefore the same for all chords through the highest point of the circle.

(8) The times required by heavy particles to descend in straight lines to the lowest point in the circumference of a vertical circle from all other points in the circumference are the same.

(9) If any focal chord PQ of a parabola be vertical, and the tangents TP, TQ be drawn, heavy particles starting simultaneously from rest at P and T, and falling along the lines PQ, TQ respectively, will reach Q at the same instant.

(10) A number of heavy particles start without velocity from a common position and slide down straight lines in various directions. Show that the locus of the points reached by them with a given speed is a horizontal plane, and that of the points reached by them in a given time is a sphere whose highest point is the starting point.

(11) Show that, if a circle be drawn touching a horizontal straight line in a point P and a given curve in a point Q (P and the curve being in the same vertical plane and P being higher than Q), PQ is the line of quickest descent to the curve (*i.e.*, a heavy particle requires less time to fall from P to the curve along this line than along any other).

(12) Find the straight line of quickest descent from a given point to a given straight line, the point and the line being in the same vertical plane.

Ans. From P, the given point, draw a horizontal line meeting AB, the given line (A being higher than B), in C. From CB cut off CD equal to CP. PD is the required line.

(13) Show that if, from a given point in the same plane as a given vertical circle and outside it, a straight line be drawn to the lowest point of the circle, the part intercepted between the given point and the circle is the line of quickest descent from the one to the other.

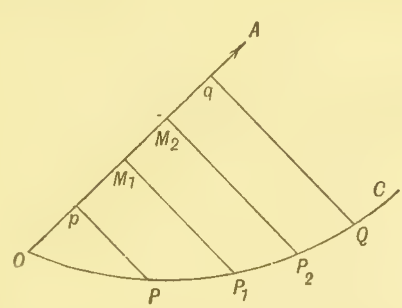
(14) Find the loci of points (a) inside, and (b) outside, a given vertical circle, which are such that the times of falling from them down lines of quickest descent to the given circle may have a given value.

Ans. (a) a circle touching the given circle at the highest point, (b) a circle touching the given circle at the lowest point.

(15) A given point P is in the same plane with a given vertical circle and outside it, the highest point Q of the circle being lower than P. Find the line of slowest descent from P to the circle.

Ans. Join PQ and produce it to meet the circumference in R. PR is the required line.

185. (2) *Motion in a Curved Path under a Uniform Acceleration.*—Let OC be the curved path and OA the direction of the acceleration a. Since any small portion P_1P_2 of the curve may be considered to be a straight line, the change of speed of the moving point between P_1 and P_2 is (181) the same as it would have been had the point moved from M_1 to M_2,



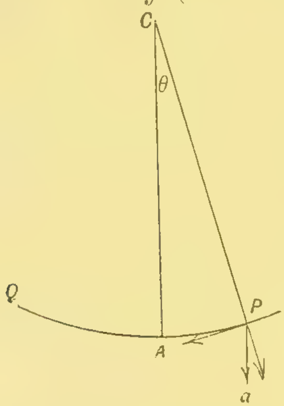
M_1M_2 being the projection of P_1P_2 on OA. Hence also the change of speed which the moving point undergoes in traversing a finite portion of its path PQ is the same as it would undergo in traversing pq, the projection

of PQ on a line in the direction of the acceleration. Hence, if V is the speed at P and v that at Q,

$$v^2 - V^2 = 2a \,.\, pq.$$

186. If a point moving under a uniform acceleration is constrained to remain on a surface, it must move in a plane curve which is the section of that surface by a plane through the position of the point at any instant and containing the directions of the acceleration and of the velocity at that instant.

187. (3) *Motion of a Point constrained to remain on a Spherical Surface under a Uniform Acceleration.*—This is the case of the *Simple Pendulum*, which consists of a small body (called the bob) attached to a fixed point by an inextensible string.—Let C be the centre of the spherical surface, CA the radius whose direction is that of the acceleration of the moving point. Let P be any position of the point. P's acceleration a may be resolved into two rectangular components in the plane PCA, one, $a\cos\theta$ (angle $ACP = \theta$), normal to the spherical surface at P, and the other, $a\sin\theta$, tangential to it and towards A. the normal component cannot affect the motion in the spherical surface. The motion of P therefore depends upon the other.



If P's velocity at any instant is wholly in the plane PCA, its acceleration being also wholly in that plane, its path must be the circle PAQ. How it will move in that path in general it is difficult to determine. But the problem is easily solved for the special case in which during the whole motion, θ is so small that it may be considered equal to $\sin\theta$. In that case P's tangential acceleration is $a\theta$, or if the length of CA be l, $a \times \text{arc } AP/l$.

It is therefore directly proportional to the displacement of P from A (measured along the path). P's motion is consequently simple harmonic, about A as centre (164). The period of the motion is thus (163)

$$2\pi\sqrt{\text{displacement} \div \text{tangential acceleration}}.$$

For a displacement of arc AP the tangential acceleration is $a \times \text{arc } AP/l$. Hence, if T is the period,

$$T = 2\pi\sqrt{\frac{l}{a}},$$

and is independent of the amplitude.

The time of oscillation of a pendulum swinging in a vertical plane is usually taken to be half the period, *i.e.*, to be the time between the instants at which the pendulum reaches opposite ends of its oscillation. Thus the seconds' pendulum is one making a complete oscillation in 2 seconds.

If θ is not indefinitely small, $\sin\theta$ is less than θ. The tangential acceleration therefore increases less rapidly than the displacement; and the period of the oscillation, which will be approximately simple harmonic if θ is comparatively small, will increase with θ.

188. If P's velocity at any instant is not wholly in the plane PCA, it may be resolved into two rectangular components, as in 167, both tangential to the spherical surface. Hence, in the case in which θ is indefinitely small, P's motion may be resolved into two simple harmonic motions of the same period; and its motion is therefore (174) elliptic harmonic motion, the period being the common period of the components, the particular ellipse described being dependent upon the amplitude and epoch of the components, and therefore upon the magnitude and direction of the point's initial velocity.

189. If θ is not indefinitely small, and if the component motions are of different amplitudes, the periods will have different values. If they are very nearly equal, the point P (*i.e.*, the pendulum bob) will go through the motions described in 179.

190. In the case in which the component motions are equal in amplitude, and therefore in period, and differ in phase by one quarter period, the point P will move (173) in a circle about the foot of the perpendicular on CA (187), as centre. This is the case of the conical pendulum (320, Ex. 19).

191. *Examples.*

(1) Find the time of oscillation of a pendulum 20 ft. long at a place at which $g=32{\cdot}2$ ft.-sec. units.

Ans. 2·47... sec.

(2) Find the length of the seconds' pendulum at a place at which $g=31{\cdot}9$ ft.-sec. units.

Ans. 3·232... ft.

(3) Find the length of the pendulum which makes 24 beats in 1 min. where $g=32{\cdot}2$. ft.-sec. units.

Ans. 20·39... ft.

(4) A seconds' pendulum is lengthened 1 per cent. How much does it lose per day?

Ans. 7 min. 8·8... sec.

(5) The length of the seconds' pendulum being 99·414 cm., find the value of g.

Ans. 981·17 cm.-sec. units.

(6) A pendulum 37·8 inches long makes 182 beats in 3 min. Find the value of g.

Ans. 31·78... ft.-sec. units.

(7) If two pendulums at the same place make 25 and 30 oscillations respectively in 1 sec., what are their relative lengths?

Ans. 1·44 : 1.

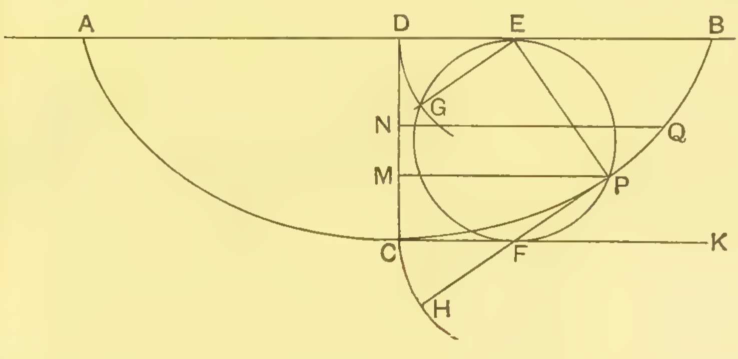
(8) A pendulum which beats seconds at one place is carried to another where it gains 2 sec. per day. Compare the values of g at these places.

Ans. As 0·99995... : 1.

(9) A pendulum which beats seconds at the sea-level is carried to the top of a mountain where it loses 40·1 sec. per day. Assuming the value of g to be inversely proportional to the square of the distance from the centre of the earth, and the sea-level to be 4,000 miles from that point, find the height of the mountain.

Ans. 1·86 miles.

192. (4) *Motion of a point constrained to remain on a cycloid, the acceleration being uniform, in the direction of the axis, and from the base towards the vertex.*—The cycloid is a curve traced by a point fixed in the circumference of a circle which rolls in its own plane along a straight line. If the circle GP roll along the line AB, starting from the position in which the tracing-point P is at A, P's path will be the cycloid ACB. If C is the position of P when the diameter of the circle through P is perpendicular to AB, CD (perpendicular to AB) is called the axis of the cycloid, and the point C its vertex.



During the motion of the rolling circle from the position in which P is at C to any such position as that shown in the diagram, P and the diametrically opposite

point G describe similar and equal cycloids of which CP and DG are portions. PEG is clearly a right angle. If therefore PF be drawn parallel to EG, EF is a diameter of the rolling circle, and F lies in a straight line CK parallel to AB. Hence $CF=DE$. Produce PF to H making $FH=PF=EG$. Evidently the curve CH which is the locus of H, is similar and equal to DG, and is therefore a cycloid similar and equal to CB. But PF is perpendicular to EP, and, as the rolling circle, in the position shown, is turning about E, the tracing point P is moving perpendicularly to EP and therefore in the direction of the line FP. Hence PF touches the cycloid ACB at P. Similarly EG is perpendicular to the cycloid DG at G, and therefore PH is perpendicular to CH at H. Hence CH, which has been shown to be a cycloid equal and similar to CPB, is the curve which would be described by the free end of a string of the same length as the curve CPB, which had one end fixed at B and was stretched along the curve CPB, if it were unwound in the plane of the curve—a result which is expressed by the statement that the involute of a cycloid, whose starting point is its vertex, is an equal and similar cycloid. Also as FH is obviously equal to EG, $PH=2PF$, and therefore $CP=2PF$.*

Let the point which is given as moving on the cycloid ACB have, when in the position Q, a speed zero. Its speed v, when at P, will (185) be such that, a being the given acceleration, $v^2=2a.NM$, QN and PM being perpendicular to CD. Let t be the time in which the point would, with the same acceleration and with initial speed zero, move from D to C. Then $CD=\frac{1}{2}at^2$. Hence

$$v^2=\frac{4}{t^2}.NM.CD=\frac{4}{t^2}CD(CN-CM).$$

Now it is obvious from the above that

$$CP^2=4FP^2=4CM.CD.$$

* Thomson and Tait's "Elements," p. 35.

Similarly $$CQ^2 = 4CN \,.\, CD.$$

Hence $$v^2 = \frac{1}{t^2}(CQ^2 - CP^2).$$

Now t^2 being equal to $2CD/a$ is constant. Hence (163-4) the motion of the moving point in the cycloid is simple harmonic, its tangential acceleration or rate of change of speed a', being such that $1/t^2 = a'/s$, where s is its distance from C, measured along the curve. If T is the period of its oscillation (double),

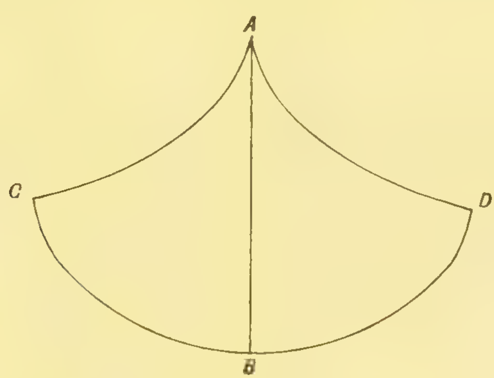
$$T = 2\pi\sqrt{\frac{s}{a'}} = 2\pi t = 2\pi\sqrt{\frac{2CD}{a}}.$$

If t' is the time occupied in moving from rest from any point Q to C,

$$t' = \frac{\pi}{2}\sqrt{\frac{2CD}{a}}.$$

As this expression involves only constants the time is the same whatever the position of the starting point may be. The cycloid is for this reason called a *tautochrone.*

193. This result is rendered of practical importance by the property of the cycloid established above, viz., that



the involute of a cycloid with its starting-point at the vertex is itself a cycloid. If therefore AC and AD are

fixed semi-cycloidal cheeks, symmetrically placed in a vertical plane about a vertical line through A, and if AB is a simple pendulum, the bob B in oscillating in the plane ACD will describe a cycloid, and its oscillations will consequently be isochronous, whatever their extent.

CHAPTER V.

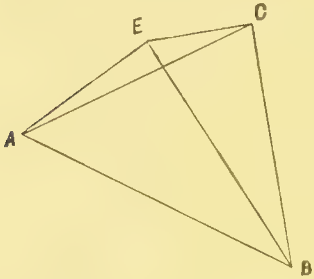
ROTATION.

194. We have called a translation any motion of a body which is such that its various points move through equal distances in the same direction. If then one point of the body be fixed, there can be no translation. The motion of which it is capable under these circumstances will be more or less complex according as its parts can or cannot move relatively to one another. We restrict ourselves here to the case of bodies whose parts cannot move relatively to one another, or, as they are called, rigid bodies. The motion of which a rigid body or system of points is capable, when one point is fixed, is called rotation.

195. It will be evident that, whatever may be the motion of such a system, straight lines through given points of the system must remain straight lines of unchanged length and inclination, and planes must continue to be planes of unchanged form, area, and inclination. It will also be evident that the motions of two points which are indefinitely near must be indefinitely nearly the same.

196. The positions of all the points of a rigid system whose configuration is known, are determined, if the

positions are known of any three points which do not lie in the same straight line. For let the positions of three points, A, B, and C be known. Then that of their plane is known also; and consequently the position of any fourth point E is known, for it must maintain given fixed distances from this plane and from every point in it, and must remain always on the same side of the plane.

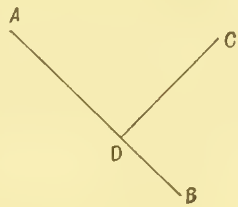


197. If therefore one point A of a rigid system is fixed, the specification of the positions of two other points B and C, not in the same straight line with the fixed point, determines the position of the system. Now A's position being fixed, and the distance of B from A being given, B's position is known, if the direction of AB is known. And the direction of AB can be described (3) by a statement of the magnitudes of two angles. Hence, A's position being given, B's position is determined by two numbers. B's position being given, and C's distance from B, C's position is known if the direction of BC is known. Now to determine this direction one angle, viz., ABC, is already known (the three sides of the triangle ABC being known). Hence one other angle determines C's position. Hence, also, the positions of A and B being given, one number determines that of C. If therefore the position of one point of a rigid system is fixed, the positions of any other two points not in the same straight line with the first, and therefore the position of the rigid system itself, are determined by three numbers.

198. *Degrees of Freedom.*—The position of a rigid system with one point fixed being described by three numbers, any change of position will be described by the changes which these numbers undergo. Any motion of a rigid system, one point of which is fixed, may therefore

be specified by three numbers; and such a system is consequently said to have three degrees of freedom.

199. *Rotations.*—If a line passing through the fixed point of a rigid system be also fixed both in the system and in space, the various points of the body can move only in circular arcs, these arcs being in planes perpendicular to the given fixed line, and their centres being the intersections with the fixed line, of perpendiculars on it from the various points. Thus, if A be the fixed point, and AB a fixed line of the system, any point C can move only in a plane through C perpendicular to AB, and its path must be a circular arc whose centre is D the foot of the perpendicular from C on AB, and whose radius is DC.



The angle between the final and initial positions of DC is (126) the angular displacement of C about AB. As all planes of a rigid system must remain planes, and must maintain their mutual inclination, the angular displacements of all the points of the system about AB must be the same as that of C. This angular displacement is therefore called the angular displacement of the system about AB.

Even if the line AB fixed in the system be not fixed in space, a motion of the system may be specified by reference to AB, and in that case also the various points of the system must move in circular arcs relatively to AB, though relatively to a line fixed in space they may have a much more complex motion.

The motion of a rigid system with one point fixed about a line through that point and fixed in the system is called a rotation of the system about the fixed line and the fixed line is called the axis of the rotation. A rotation is thus completely specified if the direction of

the fixed line is given, the sense of the rotation about it and the magnitude of the angular displacement. It is thus a vector, and may be completely represented by a straight line, whose length is proportional to the angular displacement and whose direction is such as to indicate both the axis and the sense of the rotation about it. For this purpose the line is so drawn from the fixed point, that the perpendicular from any point of the system on the axis, will seem to an observer looking along it towards the fixed point, to move counter-clockwise.

200. *Composition of Successive Rotations.* A rigid body with one point fixed undergoes successive rotations: it is required to determine the resultant rotation. The given rotations may be about the same or about different axes.

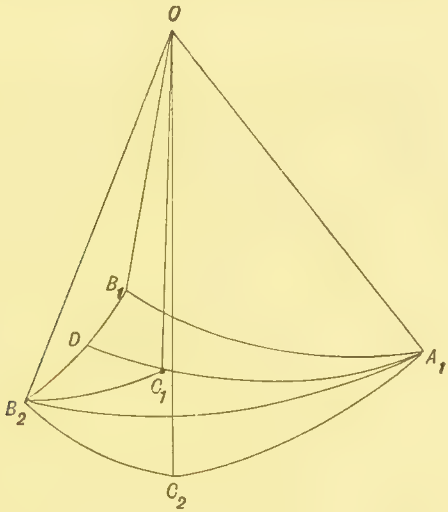
(*a*) *About the same axis.* It is obvious that the resultant of any number of successive rotations about the same axis is equal to their algebraic sum.

It follows that any rotation about a given axis may be broken up into any number of successive rotations about the same axis, provided the algebraic sum of their magnitudes is equal to the magnitude of the given rotation.

201. (*b*) *About different axes.* As these axes must pass through the fixed point they must be inclined to each other. The angular displacements about them may be finite or indefinitely small.

First, let the rotations be finite. Let O be the fixed point of the system, OA and OB, drawn so as to indicate the sense of the rotation, the axes fixed in the system about which the rotations occur, and θ and ϕ the magnitudes of these rotations respectively. Make OA equal to OB. Then during the motion A and B move on the surface of a sphere. Let OA_1, OB_1 be the initial positions

of OA and OB in space, and let OB_2 be the position in space of OB after the rotation about OA, the rotation about OA occurring first. Join A_1B_1, A_1B_2, B_2B_1 by great circles of the sphere. Then angle $B_1A_1B_2 = \theta$. Bisect this angle by a great circle meeting B_2B_1 in D. Draw a

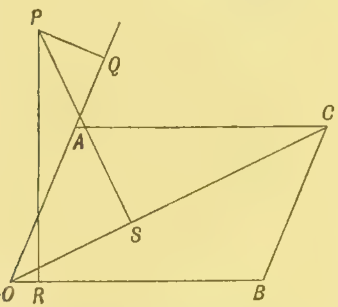


great circle through B_2, inclined to B_2A_1 at the angle $\phi/2$ and meeting A_1D in the point C_1. It is obvious from the symmetry of the sphere about a plane through its centre that a point C_2 can be found on the other side of B_2A_1 from C_1, whose position is such that $B_2C_1 = B_2C_2$, $A_1C_1 = A_1C_2$, angle $A_1B_2C_2 = \phi/2$ and angle $B_2A_1C_2 = \theta/2$.

If now the system be rotated about OA, OB will move from the position OB_1 to OB_2 and the line OC of the system initially occupying the position OC_1 in space will come to occupy the position OC_2. When now the system is rotated about OB in its new position OB_2, OC must move from the position OC_2 to the position OC_1, for the angle $C_2B_2C_1$ is equal to ϕ and $B_2C_2 = B_2C_1$. Hence the line OC fixed in the system has the same position in space after the rotations as before them; and therefore the resultant motion is a rotation about OC.

If the successive rotations occur in the reverse order, they will be found in that case also to give as resultant a single rotation about a determinate axis. But it will be obvious, especially if a construction similar to the above be made, that the resultant rotations in the two cases are in general about different axes.

202. Secondly, let the rotations be indefinitely small. Let them be represented by the lines OA, OB. Complete the parallelogram AB and join OC. Take P any point of the rigid system in the plane of OA and OB and outside the angle AOB, and draw PQ, PR, PS perpendicular to OA, OB, OC respectively. If the rotation OA occur first, P will move perpendicularly to the plane of OA and OB towards the reader through an indefinitely small distance represented by $OA \times PQ$ (199). When the rotation OB occurs, P will move in a direction perpendicular to the plane of OA and OB, in its slightly displaced position, towards the reader through a distance represented by $OB \times PR$. As these linear displacements have indefinitely nearly the same direction, the resultant displacement of P is (86, III.; and 105, footnote)



$$OA \,.\, PQ + OB \,.\, PR = OC \,.\, PS.$$

If P is a point on OC, $OA \,.\, PQ$ and $OB \,.\, PR$ are equal and of opposite sign. The resultant linear displacements of all points in OC are therefore zero, and the resultant motion is (199) a rotation about OC. As all points of the system must have the same angular displacement about OC, the magnitude of the displacement may be determined by finding it in the case of any one point. As seen above, the resultant linear displacement of a point P in the plane of OA and OB is represented by $OC \,.\, PS$. Its angular displacement about OC is therefore represented by OC.

Hence the resultant of two indefinitely small rotations represented by OA and OB, occurring in this order, is represented by OC.

The result would clearly be the same if the rotation OB occurred first, and whether OA and OB were axes fixed in the body or fixed in space.

Hence, if a rigid system with one point fixed undergo two successive indefinitely small rotations about different axes either fixed in the system or fixed in space, and if these rotations are represented by two adjacent sides of a parallelogram, the resultant displacement will be a rotation represented by the diagonal of the parallelogram through their point of intersection.

If there are more than two successive indefinitely small rotations, the third may be compounded with the resultant of the first two by the above parallelogram law, and so on.

203. *Composition of Simultaneous Rotations.*—Simultaneous rotations are usually called component rotations.

(*a*) *About the same axis.*—Let the component rotations α, β, γ, etc., be broken up each into n rotations about the same axis and of the magnitudes α_1, α_2, etc., β_1, β_2, etc., γ_1, γ_2, etc., where n is a large number; and let these rotations occur in the order α_1, β_1, γ_1, etc., α_2, β_2, γ_2, etc., and so on. If n is indefinitely great this is equivalent to the simultaneous occurrence of α, β, γ, etc. And the resultant of all these rotations is obviously (200) equal to $\alpha+\beta+\gamma+$ etc. Hence the resultant of any number of component rotations about the same axis is their algebraic sum.

204. (*b*) *About different axes.*—Let OA, OB (Fig. of 202) represent two component finite rotations about axes either fixed in the system or fixed in space. Let them be broken up each into n equal indefinitely small rotations of the magnitudes OA/n, OB/n respectively. Let

these rotations occur in the order OA/n, OB/n, OA/n, OB/n, and so on. This is equivalent to the simultaneous occurrence of OA and OB. Then (202) the resultant of the first pair of small rotations is a rotation OC/n, that of the second pair the same, and so on for the n pairs. Hence the resultant of all is equal to n rotations of the magnitude OC/n each and about the axis OC; that is (200), it is a rotation OC.

It will be noticed that this result is based on the assumption that equal fractions of the simultaneous rotations occur in equal times. Hence two component rotations which are such that equal fractions of them occur in equal times may be compounded according to the parallelogram law. It was not necessary to make this restriction in the case of component translations (see 78) because two successive translations, even if finite, have as resultant a translation which is independent of the order of their occurrence.

205. If there are more than two such components, the third may be compounded with the resultant of the first two, and so on.

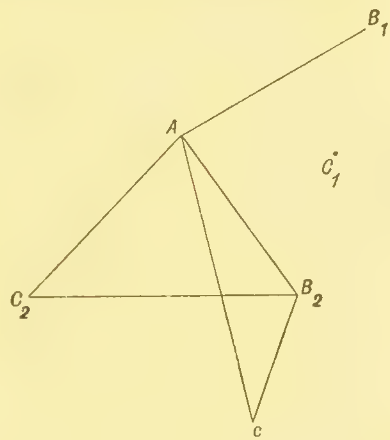
206. Component rotations of the above type are thus compounded according to the same law as component translational displacements. We have therefore propositions called the triangle, the parallelogram, and the polygon of rotations, the enunciation of which may be left to the reader. It follows that all the formulae of 85–90, deduced from similar propositions, apply to rotations of the type mentioned as well as to translations.

207. *Resolution of Rotations.*—It follows also that rotations may be resolved into components of this type after the same manner as translations.

208. *Rotational Displacements in general.*—In any displacement of a rigid system with one point fixed, there

is one line fixed in the system which has the same position in space in both the initial and the final positions of the system.

Let A be the fixed point of the system, and B and C other two points not in the same straight line with it. Let B_1, C_1 be the initial positions, and B_2, C_2 the final positions in space of B and C respectively.



As the system is rigid we must have $AB_1 = AB_2$. Hence the point B may be brought from B_1 to B_2 by a rotation about an axis through A perpendicular to the plane of B_1AB_2. By this rotation C will be moved from C_1 to a new position c, which, owing to the rigidity of the system, must be such that $Ac = AC_2$ and $B_2c = B_2C_2$. Hence the triangle AB_2C_2 is equal in all respects to the triangle AB_2c, and therefore c may be brought to coincide with C_2 by a rotation about AB in the position AB_2. Hence the given displacement may be produced by two successive rotations about axes passing through the fixed point A. Now (201) two such rotations give as resultant a single rotation about an axis through A. Hence the given displacement may be produced by a single rotation about an axis through A, and this axis has therefore the same position in space in both the initial and the final positions of the system. This axis is called the axis of the displacement.

209. Hence any displacement of a rigid system with one point fixed may be completely specified by giving the direction of its axis and the magnitude of the angular displacement about the axis.

210. Hence the three numbers which determine any displacement of such a system (198) may consist of two determining the direction of the axis and one giving the magnitude of the angular displacement about that axis: and therefore the three degrees of freedom of such a system consist of freedom to rotate about any axis through the fixed point.

211. It follows from 208 and 207 that any displacement of a rigid system with one point fixed may be resolved into three rotations about given rectangular axes through the fixed point. Any such displacement may therefore be completely determined by three numbers which are the magnitudes of rotations about three given rectangular axes. Hence the three degrees of freedom of a rigid system with one point fixed are usually described as consisting of freedom to rotate about each of three rectangular axes.

212. *Angular Velocity of a Rigid System.*—The *mean angular velocity* of a rigid system with one point fixed, during a given time, is a quantity whose magnitude is the angular displacement produced during that time, divided by the time, and whose direction is that of the axis of the angular displacement.

In general both the direction and the magnitude of the mean angular velocity of a rotating body vary with the interval of time over which it extends. If both direction and magnitude are the same whatever the interval of time, the rotation and the angular velocity are said to be uniform. In that case the axis of rotation is constant in direction and the angular displacement is proportional to the time.

213. The *instantaneous angular velocity* of such a system at a given instant has a magnitude and a direction which are the limiting magnitude and the limiting direction of the mean angular velocity between that instant

and another when the interval of time between them is made indefinitely small. The direction of the instantaneous velocity is called the instantaneous axis of rotation.

In the case of bodies under finite forces (295) the instantaneous angular velocity, as above defined, has always a finite value, and abrupt changes of the direction of the axis are impossible.

The angular velocity of a rigid system (whether mean or instantaneous) is thus seen to be a vector. It may be represented by a straight line after the same manner as a rotation.

214. *Relation between the Angular Velocity of a Rigid System and the Linear Velocity of one of its points.*—As all points of a rigid system with one point fixed move, at least instantaneously, in circular paths about the axis of rotation, the linear velocity of a point (130) will be equal to the product of its angular velocity into its distance from the axis. If ω is the angular velocity of the system and v the linear velocity of a point whose distance from the axis is r, we have $v = \omega r$.

215. The angular velocity of a system is measured in terms of the same unit as the angular velocity of a point (128).

216. *Composition of Angular Velocities.*—If a rigid system with one point fixed have any number of component angular velocities of given magnitudes and directions, we may prove, by reasoning similar to that employed in determining the law of the composition of linear velocities, that their resultant is to be determined according to the same law. We have therefore propositions called the parallelogram, the triangle, and the polygon of angular velocities of the same form as the similar propositions for linear velocities. The reader can easily formulate them for himself.

217. All the deductions from these propositions made in the case of linear velocities may also be made in that of angular velocities; and hence all the formulae of 85-90 apply to angular velocities, d_1, d_2, etc., being now the magnitudes of the component angular velocities and R the magnitude of the resultant.

218. It follows also that angular velocities may be resolved after the same manner as linear velocities or displacements (79–84).

219. *Angular Acceleration of a Rigid System.*—The angular velocity of a rigid system will in general vary from instant to instant both in magnitude and direction.

The *integral angular acceleration* of a rigid system during any time is that angular velocity which must be compounded with the initial angular velocity, in order to produce the final angular velocity.

The *mean angular acceleration* of such a system during any time has a direction which is that of the integral angular acceleration, and a magnitude which is that of the integral angular acceleration divided by the time. In general the mean angular acceleration will be different for different intervals of time. If it is the same, both in magnitude and in direction, whatever be the interval of time, the system is said to be rotating with uniform angular acceleration.

The *instantaneous angular acceleration* of a rigid system at a given instant has a direction and a magnitude which are the limiting direction and the limiting magnitude of the mean angular acceleration between that instant and another when the interval of time between them is made indefinitely small. The instantaneous angular acceleration of a rigid body is in all cases finite.

220. The angular acceleration of a system is measured in terms of the same unit as that of a point (136).

221. *Composition and Resolution of Angular Accelerations.*—The laws of the composition and resolution of angular accelerations are the counterpart of those of linear accelerations. As the latter were deduced from the laws of the composition of linear velocities, so may the former be deduced from the laws of the composition of angular velocities.

222. It follows that the relations between the magnitudes and inclinations of the components and the magnitude and direction of the resultant, as expressed in the formulae of 85–90, hold also for angular accelerations, d_1, d_2, etc., standing now for the magnitudes of these accelerations.

223. If a rigid body have an angular velocity and an angular acceleration about different axes, the acceleration may be resolved into components, one of which has the same direction as the velocity, the other a direction perpendicular to it. The effect of the former will obviously be to increase the magnitude of the angular velocity.

The effect of the latter is not so obvious. Let ω_1 be the angular velocity of the body at any instant, and let ω_2 be the angular velocity produced in an indefinitely short time thereafter by the angular acceleration, the axis of ω_2 being thus perpendicular to that of ω_1. The resultant angular velocity will be $\Omega=(\omega_1{}^2+\omega_2{}^2)^{\frac{1}{2}}$ inclined to the axis of ω_1 at an angle θ such that $\cos\theta=\omega_1/\Omega$. As θ is indefinitely small we have $\cos\theta=1-\frac{1}{2}\theta^2$. Hence

$$\Omega=\omega_1/(1-\tfrac{1}{2}\theta^2)=\omega_1(1+\tfrac{1}{2}\theta^2).$$

Hence also $$\Omega-\omega_1=\tfrac{1}{2}\omega_1\theta^2,$$

and therefore the change in the magnitude of the angular velocity during an indefinitely short time is negligible relatively to the change in its direction. The effect of the component angular acceleration which is perpendicular to the direction of the angular velocity is thus to

change the direction but not the magnitude of the velocity.

If therefore a rotating body have an angular acceleration which is continually perpendicular to its axis of rotation, its angular velocity will change in direction but not in magnitude. If it have an acceleration which has the same direction as its angular velocity, the direction of the rotation will be constant, and the rate of change of the magnitude of its angular velocity will be equal to its angular acceleration. (See 120 for the corresponding proposition in the case of translation.)

224. *Motion under given angular accelerations*, of a rigid system with one point fixed.

(1) *Angular acceleration zero.*—If there is no acceleration, the direction and magnitude of the angular velocity must remain constant. Hence, as in 138, if ω be the angular velocity and θ the displacement in a time t, we have $\theta = \omega t$; and the axis of the angular displacement is the constant axis of rotation.

225. (2) *Angular acceleration constant in magnitude and direction.*

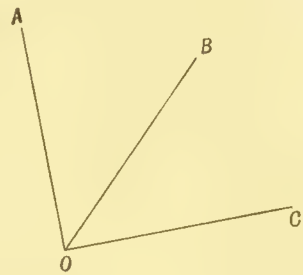
(*a*) *Direction the same as that of the instantaneous axis at any instant.*—The directions of the acceleration and of the instantaneous axis of rotation at a given instant being the same, that of the instantaneous axis is constant (140). Hence, also, the rate of change of the magnitude of the angular velocity is equal to the angular acceleration. If therefore a be the magnitude of the angular acceleration, ω_0 and ω_t the initial and final angular velocities, and θ the displacement in a time t, we may obtain, as in 63-65, the formulae—

$$\omega_t = \omega_0 + at,$$

$$\theta = \omega_0 t + \tfrac{1}{2}at^2,$$

$$\omega_t^2 = \omega_0^2 + 2a\theta.$$

226. (*b*) *Direction any whatever.*—Let OA represent in direction the initial angular velocity ω, OB that of the angular acceleration α, the angle AOB between their directions being ϕ. In the plane of OA and OB draw OC perpendicular to OA. The components of the acceleration in the directions OA and OC are thus $\alpha \cos \phi$ and $\alpha \sin \phi$.



To find the angular velocity after any time t we know that its components about OA and OC are $\omega + \alpha t \cos \phi$ and $\alpha t \sin \phi$ respectively. Hence, if Ω is its magnitude,

$$\Omega = \{(\omega + \alpha t \cos \phi)^2 + (\alpha t \sin \phi)^2\}^{\frac{1}{2}}.$$

Also, if ψ is the angle made by its direction with OC,

$$\psi = \tan^{-1} \frac{\omega + \alpha t \cos \phi}{\alpha t \sin \phi}.$$

The component angular displacements after any time t, about OA and OC will be

$$\omega t + \tfrac{1}{2}\alpha t^2 \cos \phi \quad \text{and} \quad \tfrac{1}{2}\alpha t^2 \sin \phi,$$

respectively. Their ratio is

$$(\omega + \tfrac{1}{2}\alpha t \cos \phi)/(\tfrac{1}{2}\alpha t \sin \phi),$$

and consequently varies with the time. As equal fractions of them do not therefore occur in equal times, the parallelogram law for finite displacements cannot (204) be applied. To determine their resultant the law for indefinitely small displacements must be applied by the aid of higher mathematical methods.

227. In actual practical problems the angular accelerations are usually given about axes fixed in the body, which during the motion of the body may change their directions in space. Even therefore after the axis of the resultant angular displacement has been determined, and

its magnitude, the complete solution of the problem involves in addition the determination of the direction of this axis in space. In general such determination requires the application of higher mathematical methods. We shall have to restrict ourselves consequently to cases in which the axes of the given accelerations are not only fixed in the body but have also fixed directions in space.

228. *Examples.*

(1) A rigid system has two component angular velocities, whose magnitudes are 2 and 4 radians per second respectively, and whose axes are inclined 60°. Find the resultant angular velocity.

Ans. Magnitude, $2\sqrt{7}$ radians per second; axis inclined $\sin^{-1}\frac{1}{2}\sqrt{\frac{3}{7}}$ to the greater component.

(2) A sphere with one of its superficial points fixed undergoes two component rotations—one of 8 radians about a tangent line, and one of 15 radians about a diameter, equal fractions of the components occurring in equal lines. Find the axis of the resultant displacement and the number of complete revolutions made about it.

Ans. Inclination of axis to greater component is $\tan^{-1}\frac{8}{15}$, and the number of revolutions is $17/2\pi$.

(3) A sphere is rotating uniformly about a diameter at the rate of 10 radians per min. Find (a) the component angular velocity about another diameter inclined 30° to the former, and (b) the component rotation produced in 2 min. about a diameter inclined 45° to the first.

Ans. (a) $5\sqrt{3}$ radians per min.; (b) $10\sqrt{2}$ radians.

(4) A pendulum, suspended at a point in the polar axis of the earth, oscillates in a vertical plane. Given that the plane of its oscillation does not change its aspect relative to the fixed stars, find the motion of this plane relative to the earth.

Ans. It rotates about the polar axis at the rate of one complete rotation per day.

(5) A pendulum is hung at a place of latitude λ and oscillates in a vertical plane. With the datum of Ex. (4), find (*a*) the angular velocity of the plane of the pendulum's motion relative to the earth, and (*b*) the time in which this plane will make one complete revolution at a place in latitude 60° N. [A pendulum, so mounted that the angular velocity of the plane of its motion may be observed, is called *Foucault's pendulum*, the experiment having been first made by Foucault. That the oscillation may be in a plane, the pendulum must be long and free from unnecessary constraint at the point of suspension, must have a symmetrical bob, and must be very carefully started. To obtain the angular velocity of the plane of the pendulum's motion, note that it is only the component of the earth's angular velocity about an axis through the centre of the earth and the point of suspension of the pendulum, which involves a relative motion of the plane of the pendulum's motion and the surface of the earth.]

Ans. (*a*) $2\pi \sin \lambda$ radians per day east to west; (*b*) $\dfrac{2}{\sqrt{3}}$ days.

(6) A cube rotates about a vertically upward axis through one of its edges. At a given instant at which that diagonal of the upper surface which passes through the axis, points north, the cube has an angular velocity of 40 radians per sec. and begins to have a uniform angular acceleration about an axis vertically downwards through the same edge, of 6 rad.-per-sec. per sec. (*a*) In what direction will the above diagonal point after 20 sec.? (*b*) How many revolutions will the cube have made?

Ans. (*a*) S. 57°·96... W.; (*b*) $\dfrac{333\cdot\dot{3}}{\pi}$.

(7) A sphere is rotating about its centre C with component angular accelerations of 4 and 2 rad.-per-sec. per sec. about two diameters ACB and DCE respectively inclined 60°. At a given instant its angular velocities about the same diameters are $\frac{1}{8}$ and $\frac{1}{16}$ rad. per sec. respectively. Find (*a*) the angular velocity and (*b*) the angular displacement after 20 sec.

Ans. (*a*) $\dfrac{81}{2}\sqrt{7}$ rad. per sec., and (*b*) $\sqrt{7}\left(\dfrac{9}{2}\right)^4$ rad., about an axis inclined to DCE at the angle $\sin^{-1}\sqrt{\dfrac{3}{7}}$.

229. *Geometrical representation of the motion of a rigid system about a fixed point.*—At any instant the instantaneous axis of a rotating rigid body occupies both a definite position in the body and a definite position in space; but from instant to instant it changes both its position in the body and its position in space. As its changes of direction must (295) be gradual, the successive positions of the axis in the body describe a surface in the body, and the successive positions of the axis in space describe a surface in space. One point of the body being fixed, the instantaneous axes all pass through it. Hence the surfaces, described as above, both in the body and in space are conical surfaces. At each instant these surfaces are in contact along a line which is the position both in the body and in space of the instantaneous axis at that instant. Hence the motion of a rigid body with one point fixed may be represented by the rolling of a cone fixed in the body on a cone fixed in space, the vertices of the cones being the fixed point. This mode of representing geometrically the rotation of a rigid body is of great utility in the higher departments of this subject.

CHAPTER VI.

MOTION OF RIGID SYSTEMS.

230. *Motion of Free Rigid Systems.*—We are now able to discuss the motion of rigid bodies or systems of points, having studied the two forms of motion which they are capable of undergoing. We shall first consider systems which are perfectly free to move.

231. *Degrees of Freedom.*—We have seen (197) that three numbers are necessary to determine the position of a rigid body one point of which is fixed. As three numbers are necessary to describe the position of that point, six will be necessary to determine the position of a rigid system which has no point fixed. Six conditions of constraint will be necessary to fix the system. It has six degrees of freedom.

232. *Displacement of a Rigid System.*—Any displacement of a rigid system may be produced by a translation of the system together with a rotation about any point in it.—Let A, B, C be the positions of any three points determining the initial position of the rigid system, and let A_1, B_1, C_1 be the positions of these points after any displacement. Translate the system so that A comes to occupy its final position A_1. Then B and C will take positions B_2, C_2, the ends of lines from B and C equal and

parallel to AA_1. Then A_1 is a fixed point in the system so far as the two positions $A_1B_1C_1$ and $A_1B_2C_2$ are concerned. Hence (208) the system may pass from the one to the other of these positions by rotation about an axis through this point.

233. In the special case of a rigid plane system moveable in its own plane, any displacement which is not a mere translation may be produced by rotation about some point in its plane.—Let BC, $B'C'$ be initial and final positions of the same line of the system. Bisect BB' and CC' in D and E, and draw DO and EO perpendiculars to BB' and CC'. DO and EO will in general be inclined and will meet in the point O. Join OB, OB', OC, OC'.

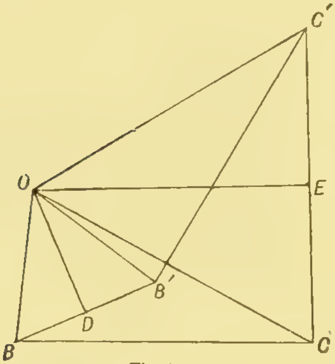


Fig 1

It may readily be shown that OB and OC are equal respectively to OB' and OC'. Also BC is equal to $B'C'$. Hence the triangle OBC is equal to the triangle $OB'C'$ in every respect, and hence OBC may be brought to coincide with $OB'C'$ by rotation about O.

Since the angle BOC is equal to the angle $B'OC'$, taking $B'OC$ from both, the remainder BOB' is equal to the remainder COC'. Hence, if the displacement is such that the point B is on CO or CO produced, the point B' will be on $C'O$ or $C'O$ produced respectively. In the former case we have Fig. 2, in the latter Fig. 3. In both these cases OD will be in the same straight line with OE, and therefore the above construction will fail. In both cases however it is obvious that the point O in which either BC and $B'C'$, or these lines produced, cut one another, is the point about which a rotation would produce the given displacement.

If, in Fig. 2, BB' be equal to CC' the point O becomes infinitely distant, and in this case also the construction

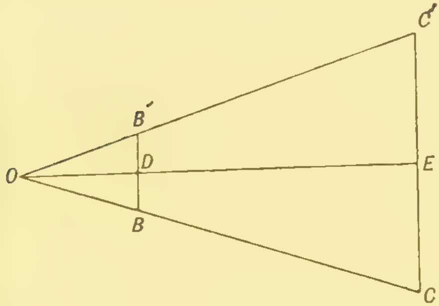


Fig 2

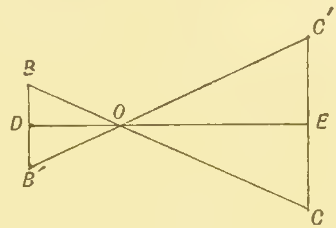


Fig 3

fails. Here however BC must be parallel to $B'C'$, and the displacement is thus a mere translation.

If BB' and CC' are indefinitely small, the point O is called the instantaneous centre of the motion of the rigid plane system.

234. It follows from 232 that the displacement of a rigid system is known if the magnitudes and directions of the linear displacement of any point in it, and of the angular displacement of the system about that point, are known.

235. Hence also the displacement of a rigid system is known if the magnitudes of the component linear displacements of any point in it parallel to three rectangular axes, and of the component angular displacements of the system about axes, parallel to the above axes, through the point, are known.

236. The six degrees of freedom of a rigid system are therefore usually described as consisting of freedom to undergo translation in, and freedom to rotate about, three directions at right angles to one another.

237. *Examples.*

(1) How many degrees of freedom has a sphere constrained (*a*) to remain in contact with a plane surface, (*b*) to remain in contact with a curved surface, (*c*) to maintain a given diameter in a given direction ?

Ans. (*a*) 5, (*b*) 5, (*c*) 4.

(2) How many conditions of constraint must be applied (*a*) to keep the surface of a sphere in contact with a given point, (*b*) to keep its centre in a given line ?

Ans. (*a*) 1, (*b*) 2.

(3) How many degrees of freedom has a three-legged stool which must remain (*a*) with the three legs in contact with a plane, (*b*) with one leg in a trench of V-shaped section, and the others in contact with a plane, (*c*) with one leg in a conical hole and the others in contact with a plane, (*d*) with one leg in a conical hole, another in a V-shaped trench, and the third in contact with a plane ?

Ans. (*a*) 3, (*b*) 2, (*c*) 1, (*d*) 0.

(4) How many degrees of freedom has a rod connected to a fixed body (*a*) by a hinge, (*b*) by a ball-and-socket joint ?

Ans. (*a*) 1, (*b*) 3.

(5) To how many degrees of constraint is a nut subjected which is moveable on a fixed screw.

Ans. 5.

(6) A nut can turn on a fixed screw. To the nut is hinged a rod on which a second screw is cut. How many degrees of freedom has a nut turning on the second screw ?

Ans. 3.

(7) A series of n rods are joined by vertical hinges, the first to a fixed body, the second to the first, and so on. How many conditions of constraint are necessary to fix the system ?

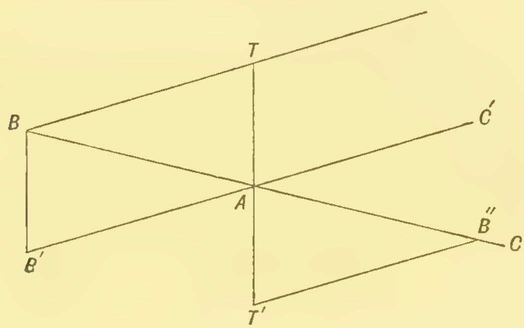
Ans. n.

(8) The ends of two rods are attached to a fixed body, their other ends to the ends of a third rod. How many degrees of freedom has the third rod if the attachments are (*a*) hinges perpendicular to the plane of the rods, (*b*) ball-and-socket joints ?

Ans. (*a*) 1, (*b*) 4.

238. *Composition of Translations and Rotations.*—We have now to determine the resultant of any number of translations and rotations which may be impressed upon a free rigid body. The following proposition will be useful.

The resultant of a rotation about a given axis and a simultaneous translation in a direction perpendicular to that axis is an equal rotation about a parallel axis.—Let the plane of the diagram be a plane of the body perpendicular to the axis of rotation, and let A be the point in



which the axis cuts the plane, and AT the direction and magnitude of the translation. Let BAC be a line of the body making with AT an angle TAB equal to half the supplement of the magnitude of the rotation. Draw $B'AC'$ so as to make the angle CAC' equal to the rotation. From T draw TB parallel to AB'; and from B, BB' parallel to TA. Then TB' is a parallelogram; and the angle TAB being equal to TAC', AB, AB', and BT are equal.

When the body is rotated through the angle CAC', the line BAC takes up the position $B'AC'$. If now it undergo the translation AT, this line moves parallel to itself so that B moves from the position B' to its initial position and A moves to T. The result of the two operations is therefore that the line BAC is brought to the position

BT, the point B occupying its initial position. As the angle ABT is equal to the angle CAC', the line BAC and therefore the whole body has undergone a rotation equal to the given rotation, about a parallel axis through B.

It will be clear that the result would have been the same had the body been first translated and then rotated, and consequently that the result would be the same were the translation and rotation simultaneous.

239. By producing TA to T', making AT' equal to AT and drawing $T'B''$ parallel to TB, it may readily be shown that reversing the direction of the component translation reverses the direction of the displacement of the axis of rotation.

240. If s is the translation and θ the rotation, we have

$$AB = \frac{s}{2\sin(\theta/2)},$$

and the angle TAB is equal to $(\pi-\theta)/2$. Hence the axis B is in a line inclined $(\pi-\theta)/2$ to the direction of the translation and at a distance from the axis A equal to $s/[2\sin(\theta/2)]$.

241. It follows from 238 that a rotation about a given axis may be resolved into an equal rotation about a parallel axis at a given distance in a given direction, together with a translation whose magnitude and direction may be determined by the above construction.

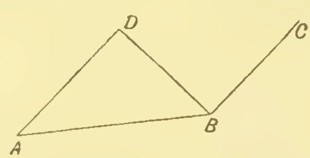
242. Hence, if a rigid system have any number of component rotations about any axes whatever (the axes will of course in general not intersect), they may be reduced to the same number of component rotations of the same magnitudes respectively, about axes parallel to the given

axes through some one point, together with as many translations. The resultant of the component rotations may then be found, in simple cases, by 206, and the resultant translation by 78.

243. Hence, if a rigid system have any number of component translations and rotations, they may be reduced to a single translation, and a single rotation about some given point. The displacement of the system may therefore be determined.

244. It is evident from 242 that the single rotation referred to in the last paragraph will have the same value whatever the position of the given point, but that the translation will vary with its position. Hence, that the displacement of a rigid system may be known, its rotation about any point fixed relatively to the system and the translation of some given point of the system must be known.

245. Every displacement of a rigid system may be produced by a rotation about a determinate axis and a translation in the direction of that axis.—Let AB and BC represent the translation and rotation to which its component translations and rotations are reducible. From A and B draw lines parallel and perpendicular respectively to BC meeting in D. Then the translation AB has the two components AD and DB. But the translation DB with the rotation about BC give as resultant an equal rotation about some other axis parallel to BC. Hence the translation AB and the rotation about BC are equivalent to a translation AD and a rotation about an axis parallel to AD.



Sir R. S. Ball has given the name *screw* to the common direction of the translation and rotation to which any

displacement of a rigid system may thus be reduced. The linear displacement in this direction per unit of angular displacement about it he calls the *pitch* of the screw. A rotation about a screw accompanied by a translation parallel to the screw through a distance equal to the product of the pitch and the angular displacement he calls a *twist* about a screw.

246. A twist about a screw is thus the most general possible motion of a rigid body. Hence one degree of constraint of the most general kind is attained by allowing a body to rotate about a given line in it, only in fixed proportion to the amount of its translation along it. Then the body has freedom to screw in the direction of this line, together with freedom to rotate about, and to be translated in, any other two directions perpendicular to one another and to the given line; on the whole, five degrees of freedom, which, with the one degree of constraint, make up the six necessary elements.

247. *Composition of Linear and Angular Velocities.*—Velocities, whether linear or angular, being displacements per unit of time, the results of 238 are true of them as well as of displacements.

In the case of instantaneous velocities however, the quantitative relations of 240 become simplified. For, if the translation and rotation of 240 are both indefinitely small, we have $AB=s/\theta$ and angle $TAB=\pi/2$. If these small displacements occur in the time τ, and if v and ω are the component instantaneous linear and angular velocities respectively, we have

$$AB=(s/\tau)\div(\theta/\tau)=v/\omega.$$

Hence the resultant of an angular velocity ω about a given axis and a linear velocity v in a plane perpendicular to the given axis is an equal angular velocity about a

parallel axis distant v/ω in a direction perpendicular to that of v.

248. Hence also a given angular velocity ω may be resolved into an equal angular velocity about a parallel axis distant d in a given direction, together with a linear velocity equal to ωd perpendicular to the plane of the axis.

249. *Composition of Angular Velocities about Parallel Axes.*—By the aid of this result, we may determine the resultant of two component angular velocities about parallel axes. Let A, B be the parallel axes, d the distance between them, ω_1, ω_2 the angular velocities about A, B respectively. Then the angular velocity ω_1 about A is equivalent to an equal angular velocity about B with a linear velocity perpendicular to the plane of the axes and equal to $\omega_1 d$. The given angular velocities about A and B are therefore equivalent to an angular velocity equal to their sum about B together with the linear velocity equal to $\omega_1 d$. Similarly, an angular velocity of $\omega_1 + \omega_2$ about B is equivalent to a linear velocity $-\omega_1 d$, with an angular velocity $\omega_1 + \omega_2$ about a parallel axis C, distant $-\omega_1 d/(\omega_1 + \omega_2)$ from B in the direction AB, and therefore $\omega_1 d/(\omega_1 + \omega_2)$ in the direction BA. Hence, as positive and negative equal translations destroy one another, the two angular velocities ω_1 and ω_2, about parallel axes A and B respectively, are equivalent to an angular velocity $\omega_1 + \omega_2$ about a parallel axis through C in the same plane as A and B, whose distance from B is $\omega_1 d/(\omega_1 + \omega_2)$, and from A $d - \omega_1 d/(\omega_1 + \omega_2)$, and which therefore intersects the line BA so that $BC : CA = \omega_1 : \omega_2$.

If the component angular velocities about A and B are equal and opposite, $\omega_1 + \omega_2 = 0$, and $\omega_1 d/(\omega_1 + \omega_2) = \infty$. The axis of the resultant angular velocity is thus at an infinite distance; in other words, the resultant velocity is a translational velocity in a direction perpendicular to the plane of A and B.

250. *Composition of Linear and Angular Accelerations.*—As in 116, it may be shown that the laws of the composition of linear and angular velocities apply also to linear and angular accelerations.

251. *Motion of a Rigid System under given Accelerations.*—The resultant linear acceleration of any point of the system and the resultant angular acceleration being known, together with the initial velocities, the displacement and the final velocities of the system may be determined by 138–180 and 224–226. In practical problems the angular accelerations are usually known about axes fixed in the body. Of such cases we must restrict ourselves (227) to those in which the axes fixed in the body have also fixed directions in space.

252. *Geometrical Representation of the Motion of a Rigid Lamina in its own Plane.*—The instantaneous centre of such a lamina occupies at any instant a definite position both in the lamina itself and in space; but from instant to instant its positions, both in the system and in space, change. By 295 the successive positions in the case of a body must be indefinitely near; and therefore the series of positions of the instantaneous centre in the system forms one curve and the series of positions in space forms another. At each instant these curves must be in contact, the points of the curves in contact being the positions of the instantaneous centre at the given instant. Hence the motion of a rigid plane system in its own plane may be geometrically represented by the rolling of a curve fixed in the system on a curve fixed in space. This conception is of great use in the treatment of some of the more difficult problems of the motion of rigid systems.

253. *Motion of Rigid Systems under Constraint.*—If two or more bodies are connected together in any way they form a system, the motion of any one member of

which, thus subjected to constraint, depends upon the position and the motions of the others. In such cases it may be required to determine the instantaneous axis of rotation of any one member, to find relations between the velocities of various members of the system, etc. We may illustrate such cases by a few examples. Readers who wish a thorough treatment of the constrained motion of rigid bodies should study works on the Kinematics of Machinery.

254. *Examples.*

(1) The line DE moves, keeping its ends in two fixed straight lines ADB, AEC. Find the instantaneous centre and the direction of motion of any point G in DE, when DE occupies any given position.

From D and E draw DF and EF perpendicular to AB and AC and meeting in F. F is the instantaneous centre (233); for indefinitely small displacements of D and E have the same directions as AB and AC respectively, and their middle points coincide ultimately with D and E. Join GF. The line through G perpendicular to GF is the direction of G's motion at the given instant.

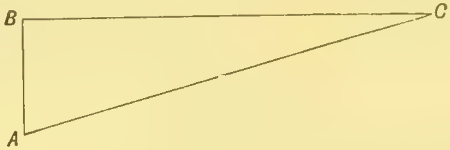
(2) A rod DE falling with its ends in contact with two other rods, one ADB vertical and the other AEC horizontal, is inclined 30° to the horizontal rod. Find (a) the direction of motion of the middle point of DE, and (b) the point of the rod whose motion is inclined 30° to AC.

Ans. (a) Inclined 60° to AC; (b) $DE/4$ from E.

(3) A rod moves so that its end points remain in a given circle. Show the centre of the circle to be the instantaneous centre of the motion.

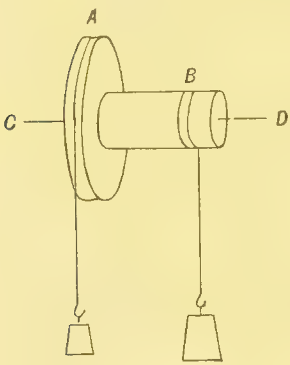
(4) Find the ratio of the velocity of any point of a screw to its velocity of advance. [The *screw* consists of a convex or concave cylinder with one or more helical projections called threads winding round it, the inclination of the thread to the axis of the cylinder being constant. The pitch of a screw with one or more threads is the distance between successive windings of the same thread measured parallel to the axis of the cylinder.]

If AB be equal to the pitch of the screw, and if BC, perpendicular to AB, be equal to the circumference of a right section of the cylinder, then the straight line AC will represent in length and inclination to AB a single winding of the thread. For only the straight line CA has its end points at C and A and is equally inclined throughout to BC and therefore to a line perpendicular to



BC. If the screw advance through the distance BA, every point in its surface will move through the distance CA; and if the screw advance through any fraction of BA, each point on its surface will move through a distance which is the same fraction of CA. Hence the ratio of the velocity of any point on its surface to its velocity of advance is CA/BA, or, if p is the pitch and r the radius of the screw, $(p^2+4\pi^2r^2)^{\frac{1}{2}}/p$.

(5) Two bodies hang by strings from the wheel and the axle respectively of the simple machine called the Wheel and Axle. Find the ratio of the magnitudes of their velocities. [The *Wheel and Axle* consists of a rigid cylinder AB, moveable about its axis CD, and having different diameters at different parts A and B, called respectively the wheel and the axle. In a simple form of it the axis is horizontal, and strings attached to the wheel and the axle respectively and wrapped round them in opposite directions, carry heavy bodies, one of which therefore rises when the other falls.]



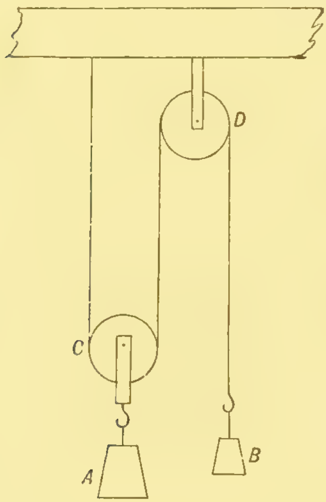
Ans. R/r, if R is the radius of the wheel and r that of the axle.

(6) Two bodies A and B are connected by means of a system of pulleys represented in the figure, one, C, being moveable, the other, D, being fixed. Express the angular velocity of C in terms of (a) the velocity v of B, and (b) the velocity v' of A. [*A pulley* is a con-

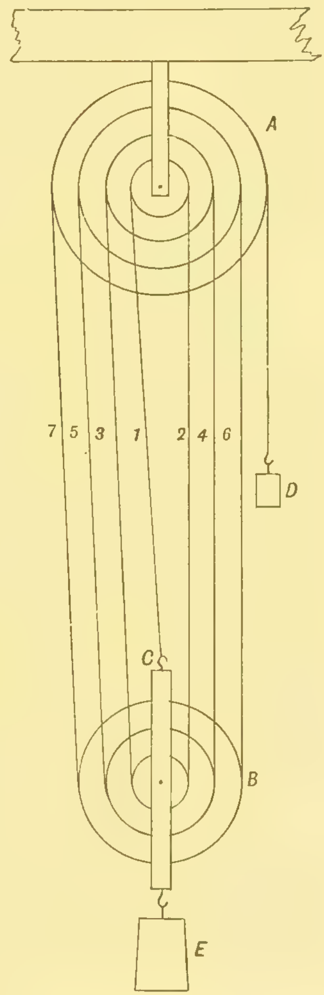
trivance for changing the direction of a string. It usually takes the form of a grooved wheel or sheaf, whose axis is fixed in a framework or block, the block being sometimes fixed, sometimes moveable.]

Ans. (*a*) $v/2r$; (*b*) v'/r, where r is the radius of C.

(7) Two bodies D and E are connected by means of the system



(Fig. of Ex. 6.)



(Fig. of Ex. 7.)

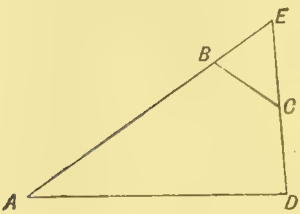
of pulleys represented in the figure, A being a fixed block with four sheaves, B a moveable block with three sheaves, and the string being fastened at C and passed round sheaves in A and B alternately until it has passed round them all. Compare the velocities of D and E, and find the radii of the sheaves that their angular velocities may be the same.

If the distance between D and A is increased by any amount, the lengths of the plies 1... 7 must be diminished each by one-seventh of that amount. Hence, if v is the velocity of D, and v' that of E, $v = 7v'$. If the sheaves have all the same radius r, they

will have different angular velocities. Let ω_1, ω_2, etc., be the angular velocities of the first, second, etc., sheaves met with in passing along the cord from C to D. Then $\omega_1=v'/r$. As twice as much cord passes round the second sheaf as round the first, $\omega_2=2v'/r$. Similarly $\omega_3=3v'/r$, and so on. That the angular velocities may be the same, the radii of the 1st, 2nd... n^{th} sheaves must be as the numbers 1, 2... n respectively.

(8) AB, BC, CD are three rigid rods jointed to one another at B and C, and to fixed points at A and D, and moveable in one plane. Find the angular velocity of CD when that of AB is ω. [The motion of this system is called *three-bar motion*. The system is one of the "elementary combinations" of machinery.]

Produce AB and DC to meet in E. Then at any instant the linear velocities of B and C are perpendicular to AB and CD respectively. Hence at that instant BC is rotating about E. Now B's linear velocity is $\omega\,.\,AB$. Hence the angular velocity of BC about E is $\omega AB/BE$. Hence also the linear velocity of C is $\omega AB\,.\,EC/BE$, and the angular velocity of CD is $\omega AB\,.\,EC/(BE\,.\,CD)$.



(9) A disc (radius $=r$) rolls without sliding on a plane. Find the relation between its angular velocity ω and the linear velocity v of its centre.

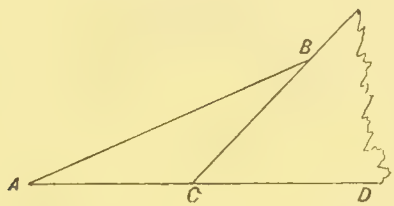
The point of the disc in contact with the plane has two component linear velocities, one the translational velocity v which it has in common with the centre, and another equal to ωr due to its rotation about the centre. These components are in the same straight line. Hence their resultant is equal to their algebraic sum. But their resultant is zero. For as the disc rolls without sliding the point of the disc in contact with the plane is instantaneously at rest. Hence $v+\omega r=0$. As this equation holds at all stages of the motion, if α and a are the angular and linear accelerations respectively, we have also $a+\alpha r=0$.

(10) A rod (length $=2l$) hangs by a small ring at its upper end from a fixed horizontal rod. To the former an angular velocity

ω is communicated in a vertical plane through the fixed rod, so that the centre of the moveable rod moves vertically. Find the linear velocity of its centre when its inclination to the vertical is θ.

Ans. $\omega l \sin\theta$.

(11) A rod AB (length $= l$) is freely moveable about a hinge at A and rests with its end B on one plane surface of a wedge BCD,



whose other plane surface is in contact with a table in which A is situated, the rod AB being in a plane perpendicular to the edge of the wedge. Show that if BCD be advanced along the table towards A, with velocity v, and if the angles BAC and BCD are θ and ϕ respectively, the angular velocity of the rod will be

$$\frac{v}{l} \cdot \frac{\sin\phi}{\cos(\phi-\theta)}.$$

CHAPTER VII.

STRAINS.

255. We have next to discuss the motion of systems of points whose distances from one another are variable. Any change of configuration of such a system is called a *strain*.

256. Strains may involve both translation and rotation, *i.e.*, there may be no point of the system which occupies the same position in space in both the initial and the final configurations of the system, and there may be no three intersecting straight lines in the system whose directions in the initial and final configurations are parallel. In considering strains however it is usual to exclude from consideration the translation involved, as occasioning no difficulty. For this purpose one point of the body is assumed to be fixed in space.

257. *Homogeneous Strains.*—We shall restrict ourselves to the most simple strains to which bodies are subjected, those, viz., which are such that the distances of pairs of points, so placed in any part of the unstrained system that the lines joining them have the same direction, are increased or diminished in the same ratio. Such strains are called homogeneous strains.

The ratio of the distance of two points after the strain to their distance before the strain is called the

ratio of the strain for the direction of the line joining them.

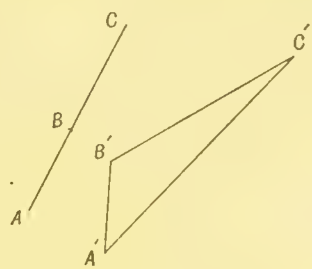
The ratio of the increment of the distance of two points to their initial distance is called the *elongation of the strain* for the direction of the line joining them. The elongations of a strain may be positive or negative.

If d and d' are the initial and final distances of two points, a the ratio of the strain, and e its elongation, for the direction of the line joining the points, we have thus

$$a = d'/d, \qquad e = (d' - d)/d.$$

Hence $$a = 1 + e.$$

258. Points which lie in straight lines before a homogeneous strain lie also in straight lines after the strain.



Let A, B, C be points lying in a straight line before the strain and let A', B', C' be their positions after the strain. Then (257)

$$A'B'/AB = B'C'/BC = A'C'/AC.$$

Hence

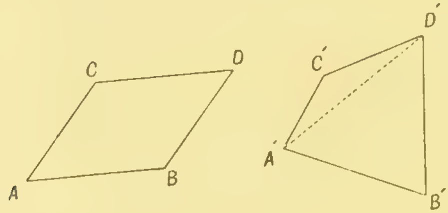
$$(A'B' + B'C')/(AB + BC) = (A'B' + B'C')/AC = A'C'/AC,$$

and hence $$A'B' + B'C' = A'C'.$$

B' is therefore a point in the straight line $A'C'$.

259. Since all straight lines remain straight after the strain it is clear that planes must remain planes.

260. Lines which are parallel in the unstrained state of the system are parallel also after the strain.—Let AB, CD and AC, BD be pairs of intersecting parallel lines in



the unstrained system. These lines being in the same plane before the strain must be in the same plane after it. Also AB and CD being equal before the strain must remain equal. Similarly AC and BD must remain equal. Moreover, CD cannot cut AB or AC cut BD after the strain; for the distance of the middle points of the intersecting lines would in that case be reduced to zero by the strain. Hence, if $A'B'D'C'$ represent the strained system, $A'C'$ and $C'D'$ are equal respectively to $D'B'$ and $B'A'$; and $A'D'$ being common to the two triangles $A'C'D'$ and $D'B'A'$, these triangles are equal in every respect. The angles $B'A'D'$ and $A'D'C'$ are therefore equal, and likewise the angles $B'D'A'$ and $D'A'C'$. Hence $A'C'$ is parallel to $B'D'$ and $A'B'$ to $C'D'$.

261. Parallel straight lines remaining parallel and straight, parallelograms must remain parallelograms, parallel planes must remain parallel, parallelopipeds must remain parallelopipeds, and figures which are similar and similarly situated must remain similar and similarly situated, after the strain.

262. Since parallel straight lines must remain parallel and must be increased or diminished in the same ratio, a circle drawn in any part of the system must be strained so that parallel chords remain parallel and become increased or diminished in length in a given ratio. Hence (173) after the strain it will be an enlarged or diminished

orthogonal projection of the circle on some plane, *i.e.*, it will be an ellipse, perpendicular diameters of the circle having become conjugate diameters of the ellipse.

There is one pair of perpendicular conjugate diameters in every ellipse, viz., the major and minor axes. Hence there is one pair of perpendicular diameters in the circle whose mutual inclination is not changed by the strain.

263. As all plane sections of a sphere are circles, a spherical portion of the unstrained system must after the strain have the shape of a figure whose plane sections are ellipses, *i.e.*, of an ellipsoid.

A cube circumscribing the sphere will become a parallelopiped (in general not rectangular) circumscribing the ellipsoid the points of contact of the cube with the sphere, which are the extremities of three diameters at right angles to one another, becoming the points of contact of the parallelopiped with the ellipsoid, which are the extremities of conjugate diameters. Hence perpendicular diameters of the sphere become conjugate diameters of the ellipsoid after the strain.

There is one set of conjugate diameters of every ellipsoid which are at right angles to one another, viz., the principal axes. One of them is the greatest diameter, another the least, and the third has in general an intermediate value. There are thus three perpendicular diameters of the sphere which after the strain become the axes of the ellipsoid. Lines in their directions in the initial configuration have the same mutual inclination in the final configuration, though the inclination of these lines to fixed lines in space or to lines in other directions in the system may have changed.

The directions of the axes of the ellipsoid in the strained system and of the corresponding rectangular diameters in the unstrained system are called the *principal axes of the strain*. The elongations in these direc-

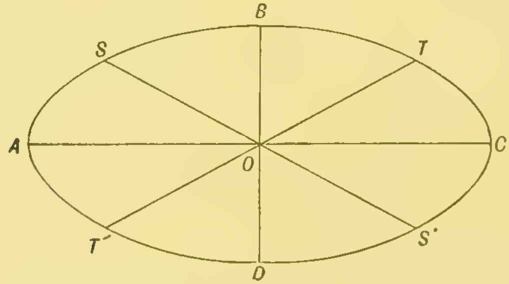
tions are called the *principal elongations*; the ratios of the strain in these directions, the *principal ratios.*

264. The ellipsoid into which any spherical portion of the system is strained is called the *strain ellipsoid.* It has obviously (257) in the case of a homogeneous strain the same form and relative position in whatever part of the system the sphere may be taken.

If the principal elongations of a strain are all equal, the strain ellipsoid becomes a sphere, and the ratios of the strain in all directions are the same as the principal ratios. All lines in a system subjected to such a strain, whatever may be their directions, are changed in length in the same ratio. There is no change of form. If two of the principal elongations are equal and the third either greater or less than the other two, the strain ellipsoid is a spheroid, prolate or oblate. If two of the principal elongations are equal to zero, it is also a prolate or oblate spheroid, its equal axes having the same length as the diameter of the sphere.

A strain in which two of the principal elongations are zero is called a *simple longitudinal strain.*

265. There are two sets of parallel planes which remain undistorted after the strain.—Let $ABCD$ be a section of the strain ellipsoid by a plane through the greatest and



least of the principal axes. Let SOS' and TOT' be diameters in this plane equal to the mean principal axis.

Then the sections of the ellipsoid by planes through SOS' and TOT' perpendicular to the plane $ABCD$ are ellipses with equal principal axes, *i.e.*, circles. Hence the elongations of all lines in these planes are the same as the mean principal elongation, and these planes therefore, and all planes parallel to them, remain undistorted after a strain though they may be changed in area. The axes AC and BD evidently bisect the angles of inclination of the planes of no distortion.

266. The ratio of the final to the initial volume of a system homogeneously strained is the same as the ratio of the volume of the strain ellipsoid to that of the corresponding sphere. Hence, if a sphere of radius r is strained into an ellipsoid whose principal semi-axes are a, b, c, the ratio of the final to the initial volume of the system is

$$\frac{(4/3)\pi abc}{(4/3)\pi r^3}=\frac{a}{r}\cdot\frac{b}{r}\cdot\frac{c}{r}=(1+e)(1+f)(1+g),$$

if e, f, and g are the respective elongations.

If e, f, and g are so small that their products may be neglected, we have

$$(1+e)(1+f)(1+g)=1+e+f+g.$$

Hence, in the case of a small strain, the *cubical dilatation*, or expansion per unit of volume, is equal to the sum of the principal elongations.

267. *Pure Strains.*—Strains in which the initial and final directions of the principal axes are the same, are called pure strains. They are so called because their characteristic property excludes the possibility of rotation.

268. *Rotational Strains.*—In general, however, the initial and final directions of the principal axes of the strain are not the same. In all such cases, since the principal axes maintain their mutual inclinations, they may be

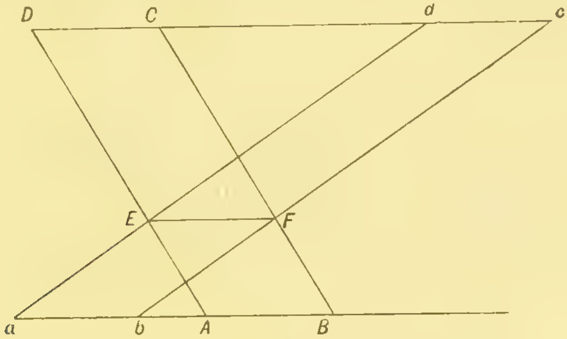
brought into the positions which they occupy after the strain, by a rotation, and the body thus rotated may then have its final configuration given it by a pure strain. It will be obvious also that the same result will be attained if the body be first subjected to the pure strain and then to the rotation.

269. *The Shear.*—If one plane of a system be held fixed, and if the planes parallel to it be moved in their own planes, without change of form or area, those on the one side of the fixed plane in any one direction, and those on the other side in the opposite direction, and all through distances proportional to their distances from the fixed plane, the system is said to have undergone a shear. The *amount* of the shear is the relative displacement of any two of the parallel planes divided by the distance between them. The *plane* of the shear is any plane intersecting the fixed plane normally in a line parallel to the direction of relative motion. The *direction* of the shear is that of the relative motion of the parallel planes.

Similarly, if one line of a plane system be held fixed, and if all lines parallel to it be moved parallel to it in one direction or the other according as they are on one side or the other of the fixed line, and through distances proportional to their distances from the fixed line, the plane system will undergo a shear, whose plane is the plane of the system, whose direction is that of the fixed line, and whose amount is the relative displacement of any two lines per unit distance between them.

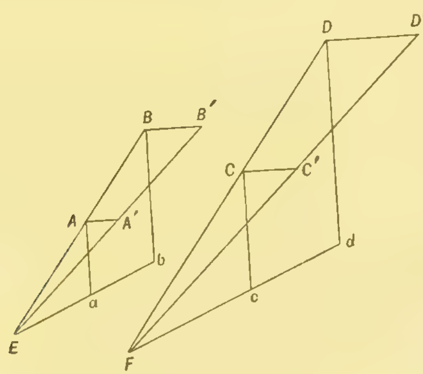
Thus any parallelogram *abcd* may be produced from any other parallelogram $ABCD$ on an equal base ($AB = ab$) and between the same parallels (aB and Dc) by subjecting it to a shear whose plane is the plane of the parallel lines aB and Dc, whose direction is that of the fixed line EF which is parallel to aB, and whose amount is the quotient of Dd by the perpendicular distance of EF from Dc.

A familiar approximate illustration of a shear in three dimensions is the change of configuration by which a



pack of cards initially forming a rectangular parallelopiped is made to take the form of a parallelopiped not rectangular. The illustration would be exact if the cards were indefinitely thin.

270. *Homogeneity of the Shear.*—Let AB and CD be parallel lines having any direction in the unstrained system. From their extremities let fall perpendiculars Aa, Bb, Cc, Dd on the fixed plane of the shear to which



the system is to be subjected. Let A', B', C', D' be the positions of A, B, C, D after the shear. Then AA', BB', CC', DD' are parallel, and

$$AA'/Aa = BB'/Bb = CC'/Cc = DD'/Dd.$$

Let BA and DC produced meet the fixed plane in E and F respectively. Then Eab and Fcd being the projections of EAB and FCD on the same plane are parallel straight lines. Since Eab is a straight line and Aa is parallel to Bb, $AE/Aa = BE/Bb$. Hence $AA'/AE = BB'/BE$; and therefore $EA'B'$ is a straight line. Similarly $FC'D'$ is a straight line. Since EB, Bb, and bE are parallel respectively to FD, Dd, and dF, the triangle EBb is similar to the triangle FDd. Hence $BE/Bb = DF/Dd$. Hence also $BB'/BE = DD'/DF$. Now BB' and BE are parallel to DD' and DF respectively. Hence the triangle EBB' is similar to FDD'; and therefore

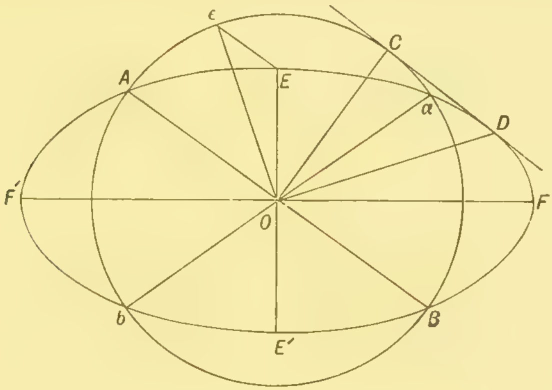
$$A'B'/AB = C'D'/CD.$$

Hence the lengths of the parallel lines AB and CD are increased by the shear in the same ratio. The shear is therefore a homogeneous strain. It has consequently principal axes, ratios, and elongations, like all homogeneous strains.

271. It is obvious that as, in a shear, all planes of a body parallel to a given plane are translated in their own planes but not changed in area, there can be no change in the volume of the body.

272. *Reduction of the Shear to a Pure Strain and a Rotation.*—Let O be the centre of a spherical portion of a system subjected to a shear, ACB the intersection of the sphere with the plane of the shear through O, and AB the intersection of the fixed plane with the same. Let the system be subjected to a shear of amount s, and such that planes parallel to the fixed plane through AB and on the C-ward side of AB move in the direction AB, parallel planes on the other side of AB moving in the opposite direction. Then (270 and 262) the circle ACB will after the shear have the form of an ellipse ADB whose centre is O; and the sphere intersecting the plane of the shear in ACB will become an ellipsoid intersecting that plane in ADB. Since the distances of points of the

system from the plane of the shear through O remain constant, this plane must contain the greatest and least principal axes of the ellipsoid. Now, by 265, EE' and



FF' the bisectors of the angles AOa and BOa respectively, will be the minor and major axes of the ellipse ADB. Hence OE, OF, and a line through O, perpendicular to both and equal to OB, are the least, greatest, and mean principal axes of the ellipsoid.

If OC is perpendicular to AB, and CD touches the circle at C, it will also touch the ellipse, and its point of contact D will be the position of C after the shear. OD is therefore conjugate to OB. Hence, since OC is equal to the perpendicular from D on OB,

$$OF/OC = OC/OE.$$

Hence the circle may be brought to have the shape of the ellipse by elongating all chords parallel to one diameter in the ratio OF/OC, and shortening all chords perpendicular to that diameter in the ratio of OC/OF.

Let a line through E parallel to OA meet the circle in ϵ. Then E is the position of ϵ after the shear. Hence (262) $O\epsilon$ and a line perpendicular to it in the plane of ACB coincide in direction after the shear with OE and OF. And a line through O perpendicular to the plane of ACB remains unchanged in direction. Hence these lines

are the positions before the shear of the principal axes of the shear. And therefore, if the sphere be first rotated about an axis through O perpendicular to the plane of ACB, through the angle ϵOE, it may then be brought to its final configuration by a pure strain whose axes are OF, OE, and a line perpendicular to both. The ratios of the strain in these axes are OF/OC, $OE/OC\,[=1/(OF/OC)]$ and 1 respectively. If OF/OC be called a, they are a, $1/a$, and 1.

273. From the symmetry of the figure it is obvious that a shear of the same plane and amount, but with the plane through ab as fixed plane, is equivalent to the same pure strain as above, together with a rotation of equal amount and about the same axis but in the opposite direction.

Hence, rotation being neglected, the same change of configuration is produced in a system by a shear of given plane and amount, whether its direction be one or other of two directions equally inclined to the greatest and least principal axes of the shear.

274. It is obvious from 265 and 272 that planes through AB and ab, normal to the plane of the shear, and all planes parallel to these planes respectively, are both undistorted and unchanged in area by each (273) of the above shears. Hence in any body subjected to a shear there are two sets of planes which are unchanged in area and form, these sets of planes being equally inclined to the greatest and least principal axes and parallel to the mean principal axis.

275. It is obvious also, with the aid of the above, that $O\epsilon$ may be brought to coincide in direction with OE, its length remaining unchanged, either by a rotation about an axis through O perpendicular to the plane ACB, through the angle ϵOE, or by a shear of the amount CD/CO in the

plane ACB, and in the direction CD, together with a pure strain whose ratios in the principal axes OF, OE have the values OC/OF and OC/OE respectively, and in a direction perpendicular to both, the value unity.

276. The amount s of the shear may be expressed in terms of its principal ratios or elongations. By a property of the ellipse (Fig. of 272)

$$OD^2+OB^2=OF^2+OE^2.$$

Hence $$OD^2=OF^2+OE^2-OC^2,$$

and $$CD^2=OF^2+OE^2-2OC^2,$$

and (272) $$CD^2=OF^2+OE^2-2OE\,.\,OF$$
$$=(OF-OE)^2.$$

Hence $$CD/OC=s=OF/OC-OE/OC=a-1/a.$$

If e is the greatest principal elongation (257),

$$s=1+e-1/(1+e).$$

If the shear be indefinitely small, we have

$$1/(1+e)=1-e,$$

and hence $$s=2e.$$

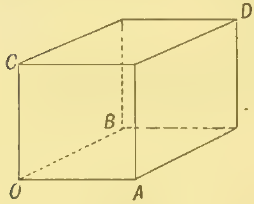
Also, when the shear is indefinitely small, OC, OD and Oa (Fig. of 272) ultimately coincide. Hence Oa is at right angles to OA, and therefore (265) is inclined to OE and OF at angles of 45°.

Hence, if a system be subjected to a strain consisting of two indefinitely small elongations, one e in any direction, and the other $-e$ in a perpendicular direction, the resulting strain is a shear whose amount is $2e$, whose plane is that of the two rectangular directions, and whose direction bisects the angle between them.

277. *Examples.*

(1) Show that, if rotation be left out of account, a small simple elongation e in any direction is equivalent to a uniform cubical

dilatation together with two shears, each having the given direction for one principal axis and lines at right angles to it and to each other for the other axes; and determine the magnitude of the dilatation and the amounts of the shears.—Let OA be the direction of the simple elongation and OD a cube of which OA is an edge. The elongation e in the direction OA is equivalent to three elongations in the same direction, each having the magnitude $e/3$. As there are no elongations in the directions OB and OC perpendicular to OA and to each other, we may regard the cube as subjected to two elongations in each of these directions, having the magnitudes $e/3$ and $-e/3$. Now an elongation $e/3$ in each of the three rectangular directions OA, OB, and OC is (266) equivalent to a uniform cubical dilatation of the magnitude e. Also, the elongation $e/3$ in the direction of OA with the elongation $-e/3$ in the direction of OB are equivalent (276) to a shear whose principal axes are these lines and whose amount is $2e/3$; and similarly the remaining elongation $e/3$ in the direction of OA with the remaining elongation $-e/3$ in the direction of OC are equivalent to a shear whose principal axes are OA and OC, and whose amount is $2e/3$.



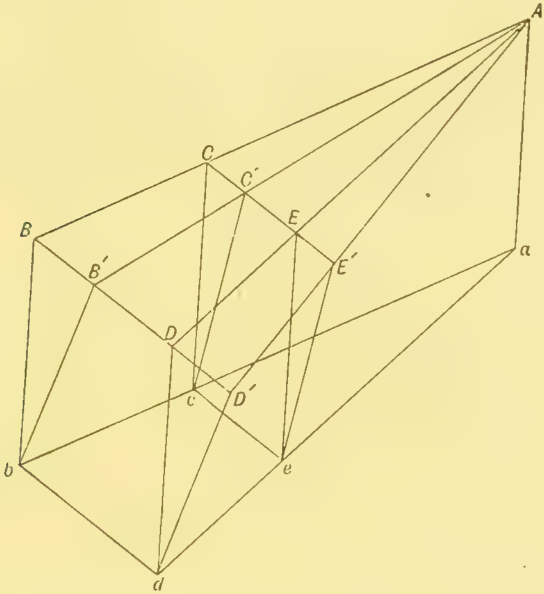
(2) Show that, if a square be subjected to a small shear whose axes are in the directions of its diagonals, it becomes a rhombus whose sides are equal to those of the square and whose angles differ from right angles by θ radians, θ being the amount of the shear.

(3) Investigate the strain in the case of a uniform circular cylinder of length l fixed at one end and having its other end twisted through an angle θ. This kind of strain is called *Torsion*.

The cylinder being uniform, every normal section of it will rotate about its axis; and, θ/l being the amount of the twist per unit length of the cylinder, the amount of the rotation of any section will be the product of θ/l into its distance from the fixed end of the cylinder. Hence, also, any normal section will be twisted relatively to any other normal section distant d from it through an angle $\theta d/l$.

Let Aa be the axis of the cylinder, $ABba$ and $ADda$ planes

through Aa, in the unstrained system, inclined at an indefinitely small angle, ABD and abd planes normal to Aa, CE and ce arcs of circles having AC as radius and A and a as centres respectively,



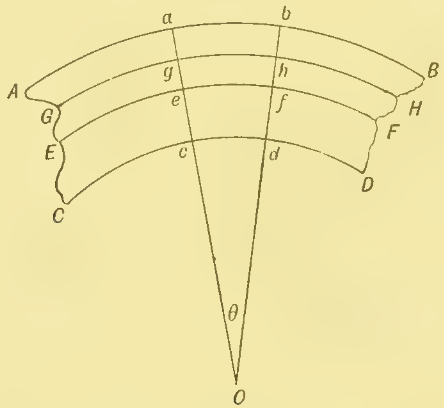
and BD and bd arcs of circles having A and a respectively as centres, and as radius AB indefinitely nearly equal to AC. Then CE, ce, BD, and bd may be considered to be equal and parallel straight lines, and $BcDe$ a rectangular parallelopiped whose edges Bb, Cc, Dd, and Ee are parallel to Aa.

After the strain B, C, D, E will have moved relatively to b, c, d, e to B', C', D', E', BB' and DD' being equal to $(\theta/l)Bb\,.\,AB$, and CC' and EE' equal to $(\theta/l)Bb\,.\,AC$. These quantities, when angle BAD and BC are made indefinitely small, are ultimately equal. Hence the small rectangular parallelopiped $BcDe$ becomes after the strain the non-rectangular parallelopiped $B'cD'e$, on the same base and between the same parallel planes. Hence the parallelopiped $BcDe$ has been subjected to a shear whose plane is $BDdb$, direction BD, and amount BB'/Bb, *i.e.*, $(\theta/l)AB$.

Hence at every point distant r from the axis of the cylinder thus subjected to torsion, it undergoes a shear whose plane is parallel to

the axis and perpendicular to a plane through the point and the axis, whose direction is normal to this plane and whose amount is $\theta r/l$.

(4) A uniform straight beam is bent so that lines initially longitudinal and straight become arcs of circles in parallel planes (called planes of bending), with centres in a line normal to these planes; transverse sections initially parallel become so inclined that they intersect in this line, and longitudinal lines in a surface, called the neutral surface, normal to planes of bending and initially a plane, are not changed in length. Investigate the strain.



Let $ABDC$ be a section of the bent beam by a plane of bending, EF the intersection with $ABDC$ of the neutral surface, ac and bd the intersections with it of two transverse sections of the beam (θ being their inclination), and O the centre of curvature of AB and CD.

Then it is obvious that longitudinal lines, such as GH, between AB and EF are lengthened, and longitudinal lines between EF and CD are shortened, by the strain. The line gh was initially equal to ef. Hence it has undergone an elongation (per unit of its length) equal to $(gh-ef)/ef$. Now $gh=Og.\theta$ and $ef=Oe.\theta$. Hence the elongation of gh is

$$(Og-Oe)/Oe=ge/Oe=d/\rho,$$

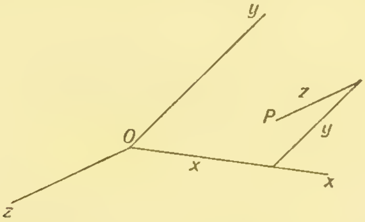
if d is the distance of the line GH from EF, and ρ the radius of curvature of EF. This result applies to all lines parallel to gh and intercepted between the transverse sections ac and bd, d being

positive when measured from ef towards ab, and negative when measured from ef towards cd. Hence at every point of the beam there is a longitudinal strain in the direction of its length, the elongation being equal to d/ρ. It is positive for all points between the convex surface and the neutral surface, and negative for all points between the neutral and the concave surfaces.

It is obvious however that these longitudinal elongations alone would not involve bending, and that in order to bring the beam into its final configuration longitudinal planes normal to planes of bending, which have thus been elongated, must slide over one another. Hence at each point of the beam there is not only a longitudinal elongation, but also a shear whose plane is the plane of bending and whose direction is longitudinal. By 273 and 276, if this shear is small it is equivalent to another in the same plane, but with a direction transverse to the beam and in the plane of bending, transverse slices of the beam sliding over one another in the direction of their intersections with planes of bending.

Hence the strain at any point of the beam consists of a longitudinal elongation equal to d/ρ, together with the above shear.

278. *Specification of a Strain.*—The elongations of a homogeneous strain in any three non-coplanar directions being given, the elongation in any other direction can be found. Let Ox, Oy, Oz be lines having any three directions and e, f, g the elongations in them respectively.



Then any point P whose co-ordinates referred to these lines as axes are x, y, z, has component displacements ex, fy, gz, and the resultant displacement may be determined by 78. The final distance of P from O may thus be determined, and hence also the elongation in the direction of OP.

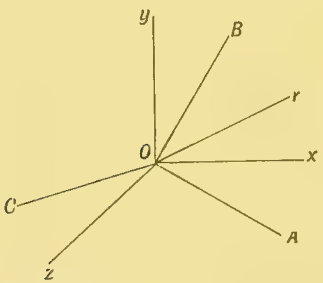
279. Hence a homogeneous strain is completely specified if the elongations in any three non-coplanar directions are given.

280. The specification of three non-coplanar directions requires (7) six numerical data, and that of the elongations in these directions three more. Hence, in general, nine quantities are requisite for the complete specification of a strain.

281. As a pure strain consists simply of elongations in certain rectangular directions, the principal axes, the specification of a pure strain requires only data sufficient to determine the directions of these axes and the elongations in them. To determine three rectangular directions three numerical data are sufficient. Hence the specification of a pure strain requires only six numerical data.

282. As any homogeneous strain may be regarded as compounded of a pure strain and a rotation, the nine data necessary for its specification may consist of the six necessary for the specification of the pure strain and the three necessary (198) for the specification of the rotation.

283. *Rectangular Specification of a small Strain.*—Let Ox, Oy, Oz be rectangular axes of co-ordinates, OA, OB, OC the principal axes of the pure strain, and Or the axis of the rotation, of which the given small strain may be regarded as compounded.



The elongations being given for the directions OA, OB, OC, equivalent elongations for the directions Ox, Oy, Oz may (278) be determined.

The rotation about Or may be resolved into component rotations about Ox, Oy, Oz. Now the rotation

about Ox being small may be regarded (275-276) as compounded of a shear whose plane is the yz plane and direction either the y or the z axis, together with a pure strain whose principal axes are Ox, a line bisecting the angle yOz, and a line perpendicular to both these. The elongation in the direction of Ox is zero, and those in the directions of the other principal axes may (278) be converted into elongations in the directions of the Oy and Oz axes. Similarly the rotation about Oy may be regarded as compounded of a shear whose plane is the xz plane and direction either the x or the z axis, together with elongations in the x and z axes; and the rotation about Oz, as compounded of a shear whose plane is the xy plane and direction either the x or the y axis, together with elongations in the x and y axes.

Now these various component strains being all small may be applied in any order. The three component elongations in the direction of the x axis are thus equivalent to a single elongation in that direction, and similarly for the components in the y and z axes respectively. Hence a small strain may be resolved into three simple elongations e, f, g in the directions of the three rectangular axes Ox, Oy, Oz respectively, and three shears whose amounts may be represented by a, b, c, whose planes are the yz, xz, and xy planes respectively, and whose directions are those of either the y or z axis, either the x or z axis, and either the x or y axis, respectively. Any small strain is therefore completely specified if the values of e, f, g, a, b, c, are given.

284. *Heterogeneous Strains.*—The elongations of a homogeneous strain we have seen to have the same values in the same directions throughout the system. In general however, in the strains to which bodies are subjected, the elongations in a given direction are different at different parts of the system. Such strains are called heterogeneous strains. If throughout the system the elonga-

tions at points indefinitely near one another are indefinitely nearly the same, the strain is said to be continuous. The strains of bodies, except in cases of fracture, are usually continuous.

The variation of the elongations from point to point being gradual in the continuous strain, they may be considered constant throughout indefinitely small spheres, and the dimensions and position of the ellipsoids into which these spheres are changed may then be determined as in the case of homogeneous strain. The ellipsoids however will in this case be different for different points of the body, and that the strain may be known, the strain ellipsoid, or sufficient data for determining it, must be known for every point of the system.

The consideration of strains of this kind requires mathematics of a higher order than readers of this work are supposed to have at command.

PART II.—DYNAMICS.

CHAPTER I.

THE LAWS OF MOTION.

285. So far we have dealt with the motion of bodies only by means of mathematical generalizations. We have thus seen how to determine the displacements, velocities, paths, of bodies when their accelerations are known. Mathematics alone does not enable us to go farther. If we wish to inquire into the way in which bodies come to have accelerations and how they influence one another in their motions, we must obtain additional generalizations on which to build; and we thus pass from the department of mathematical science into that of physical science.

Dynamics is that branch of physical science which treats of the effect of the exertion of force upon bodies.

The idea of force is ultimate. It is given us by sense. Like colour, taste, smell, it cannot be described. But we all have the idea, and when any one speaks of pushing, pulling, or exerting force in any way, we all know what he means. What the organ of the sense is from which we have this idea physiologists have not definitely settled. It has been supposed to reside in the muscles, and has been consequently called the muscular sense. We do not

require to know the seat of the sense, and may call it simply the sense of force.

A sensation of force is not qualitative merely, but quantitative as well. We recognize ourselves in any case, not only as exerting force on a body, but as exerting a greater or a smaller force, a force of definite magnitude, in a definite direction. Our power of estimating the relative magnitudes of the forces we exert is not naturally strong, but it is susceptible of cultivation; and the education of the sense of force is one of the necessary conditions of the attainment of manual skill.

According to the ordinary conception of force, the magnitude and direction of a given force are quite definite, and therefore independent of the points by reference to which motions are specified. And this is true also of much scientific usage, the gravitational attraction between two particles, *e.g.*, being always regarded as in the line joining the particles and of a magnitude depending only on their masses (288) and their distance.

A man who had no sense of force would enter upon the study of the motions of bodies without this conception. He would be restricted to the use of his eyes, and in developing the subject of dynamics might use either no conception of force or a relative conception. It would be quite possible to build up a science of dynamics without any of the data of the sense of force, just as the sciences of electricity and magnetism have been built up without electrical or magnetic senses. Some writers think it well to discard the old conception of force and to develop the subject of dynamics also by the aid of the eye alone. And the method may have scientific advantages; but from an educational standpoint it seems better to make use at the outset of all the dynamical experience that the student possesses. Later on he will be able to judge as to the advantage of reconstructing the foundations of the science on a narrower sensuous basis.

286. *First Law of Motion.*—Among our earliest generalizations are included those with regard to the effects of the exertion of force on bodies. These effects are very different in different circumstances; but when examined they are found to be in all cases composed of changes of velocity and changes of form or volume. And as a change of the form or volume of a body is a change of the relative positions and therefore of the relative velocities of its constituent parts, we find the effect of the exertion of force on bodies to be in all cases change of velocity, or acceleration.

Cases of equilibrium (323), *i.e.*, cases in which a body, though acted upon by two or more forces, has an acceleration zero, apparently form exceptions to this rule. But in such cases, if the forces are allowed to act on the body successively, the accelerations produced are found to be such as would give a resultant acceleration zero were they to occur simultaneously. Thus, though the forces together produce an acceleration zero, each may be regarded nevertheless as producing its own acceleration.

Having exerted force on all bodies within our reach and found acceleration invariably produced, we are led to expect this effect in all cases whether within the range of our experiments or not, and to conclude that a force exerted on any body will produce an acceleration in it.

We observe also, however, that many bodies move with acceleration when we are exerting no force upon them. Two billiard balls, for example, which impinge upon one another, have their velocities changed. A body which is simply let fall is found to fall with continually increasing speed. One body in short is found to be able to produce acceleration in others, it may be during contact, it may be even without contact. In such cases the effect produced is the same as if we exerted force upon the bodies; and we therefore regard the action between them as of the same kind as our action on them when we are exerting force.

We are thus led to conclude that the exertion of force on a body is invariably the antecedent of acceleration in it. We may express this result negatively by asserting that a body not acted upon by force will experience no acceleration; and it was in this form that Kepler, and afterwards Newton, enunciated it. Newton called it the first law of motion and expressed it thus—

Every body continues in its state of rest or of uniform motion in a straight line except in so far as it may be compelled by impressed forces to change that state.

The necessity of exerting force on a body in order to produce acceleration in it is often ascribed to a property of bodies called their *inertia.* That bodies have inertia, and that force must be exerted upon them in order to change their velocity, are only different modes of expressing the same experience.

The First Law is often said to furnish us with a basis for what is called the measurement of time. The recognition of uniformity of motion however presupposes such measurement. In fact the law may be enunciated without reference to the measurement of time, as follows: Bodies move through equal distances in straight lines, while the earth (more exactly an ideal earth, 30) turns through equal angles relatively to the fixed stars, except in so far as they may be compelled by impressed forces to move through unequal distances or in curved lines. That the motion of a body which is not acted upon by forces may be described thus simply, justifies us in using the earth's rotation as a standard motion, but not in asserting that the times in which it turns through equal angles are equal.

287. *Second Law of Motion.*—The First Law is a qualitative statement. We have next to ask how the acceleration produced by a force depends upon the magnitude and direction of the force.

If this question suggested itself to us now for the first

time, we should need to investigate it experimentally. For this purpose we might select as a standard force that exerted by a given spring when stretched a given amount. We might also prepare several exactly similar springs. Both our educated sense of force and our confidence in the "uniformity of nature" assure us that when extended by the given amount they exert the same force. We might act by means of these springs on, say, a curling-stone lying on a smooth horizontal surface of ice, taking care so to apply the springs that no appreciable rotation or change of form or volume should be produced. Preliminary trial would show that if, having started the stone, we exerted no force upon it, it would move with a nearly uniform speed in a straight line over the surface of the ice. We might determine by observation what the small variation of velocity was and allow for it in subsequent experiments. If now we should attach one of our stretched springs to the stone and allow it to act on the stone during known intervals of time, keeping the spring stretched to the same extent and in a constant direction as the stone moved, we could, by noting the positions of the stone at a series of instants, determine the direction and magnitude of the acceleration produced. The same determination might be made with two or with any number of springs attached and for longer or shorter periods of time. When that had been done it would be found, so far as such experiments would enable us to judge, (1) that in all cases the accelerations produced are uniform; (2) that the direction of the acceleration is always that of the force; and (3) that the acceleration produced by a force in a given body is proportional to the force, double the force producing double the acceleration, three times the force three times the acceleration, and so on. The same result would be obtained, whatever the kind or the condition of the body experimented with, whatever its initial velocity, and whatever component accelerations it might have besides that produced by the springs.

Such experiments would apply only to forces whose direction and magnitude are the same during the whole time of their action. As we should find, however, that the result does not depend upon the length of time during which the force acts, and as a variable force may be considered to consist of a succession of constant forces of different magnitudes or directions, each acting for a short time, we might extend our results to all forces, uniform or variable, and conclude that the magnitude of a variable force is at any instant proportional to the instantaneous acceleration of the body at that instant, and that their directions are the same.

As a matter of fact we have all reached the above generalizations without having had recourse to any such experiments. For in the course of our daily experience we so frequently exert forces and observe their quantitative effect, that the relation between them has revealed itself to us in a more or less precise manner. Even the small boy has found out for himself that he must push a body in the direction in which it is to be moved, and if asked how he must kick a football in order to make it move twice as fast as on a previous occasion, will answer without hesitation that he must kick it twice as hard. The real basis of our confidence in the quantitative law of force is thus not any set of experiments, but the accumulated experience of the race, including the experience of engineers, astronomers, and other scientific men, who have made elaborate calculations based on it, and always found the results borne out by fact.

288. If by F we indicate the magnitude of the force exerted on a given body, and by a that of the acceleration thereby produced, the third part of the above result may be expressed in symbols thus: $F \propto a$. Hence $F/a =$ a constant, *i.e.*, the ratio of the force acting on a given body to the acceleration thereby produced in it is constant. The value of this constant ratio will clearly depend upon

the magnitudes of the units of force and acceleration. But with given units this ratio will have a fixed value for a given body, whatever its condition (as to temperature, etc.) and whatever the circumstances of its motion.

289. In popular phraseology we speak of a body as being massive when we require to exert a great force upon it in order to produce a small change in its velocity. Thus an iron gate is said to be more massive than a wooden gate of the same dimensions, because it takes a greater force to produce in it a given angular acceleration than in the wooden gate, though the friction and other opposing forces may be the same in both cases. The greater the force required to produce a given acceleration, and the smaller the acceleration produced by a given force, in other words, the greater the value of the ratio F/a, the greater do we consider the massiveness to be.

The term massiveness is thus an appropriate designation for the ratio F/a; and if we indicate the massiveness of a body A by M, we may express the above result by the equation $F/a = M$, a constant for that body.

For any other body, B say, we have $F'/a' = M'$, a different constant. Hence

$$(F/a)/(F'/a') = M/M',$$

which we may write $\qquad F/a = km$

if k stand for the value of F/a for B, and m for the massiveness of A expressed in terms of that of B. The massiveness of any body when expressed in terms of that of another selected as standard is called its mass.* m is

* The mass of a body is by many writers defined as the quantity of matter which it contains. As we do not know what matter is, still less how to measure it, this phrase (for it is thus a mere phrase) must then itself be defined; and such writers define it more or less directly as being proportional to the ratio of the force acting on the body to the acceleration thereby produced. This mode of definition is clearly the same as that employed above, except that a useless intermediate term is introduced. The phrase quantity of inertia has been similarly used as an intermediate term.

thus the mass of the body A expressed in terms of B as standard. The value of k will clearly depend upon the standards of force and mass selected, for it represents the value of F/a for the body B.

We may accordingly express the above relation between the force acting on a body and the magnitude of the acceleration produced by it by the equation

$$F = kma,$$

m being the mass of the body, and k a constant depending only on the standards employed in specifying the values of F, m and a. This equation with the further statement that the force and the acceleration have the same direction, constitute the Second Law of Motion.

The First Law is clearly a particular case of the Second. For as no bodies of other than finite massiveness are known, if either F or a has the value zero, the other must be zero also.

290. The reader should carefully note that the massiveness of a body is quite a different conception from its heaviness. The measure of the former is called its mass, but often also its weight. The measure of the latter, viz., the force which we must balance in order to support the body, is called always its weight. The weight, in the second sense, is a force, and has different values as the body may be at different parts of, or distances from, the earth's surface. Its mass is proportional to a force divided by an acceleration, and does not vary with position.

At any one place all bodies are found to fall with the same acceleration. Now the acceleration with which a body falls is that produced in it by its weight, the downward pull on it. Let w and w' be the weights of two bodies, g the acceleration with which they fall at any given place; then their masses are proportional (289) to

w/g and w'/g respectively. If, then, m and m' are their masses, we have

$$m : m' = w/g : w'/g = w : w'.$$

Hence the masses of bodies are proportional to their weights at the same place, and the ratio of the masses of two bodies is the same as that of their weights. It is doubtless for this reason that the term weight is used for both the measure of massiveness and that of heaviness. In works in which it is used with both significations it is usually obvious from the context in any case, which is to be understood. Nevertheless, to avoid all ambiguity, we will restrict it to one, using it to denote the force which must be balanced in order that a body may be prevented from falling.

291. It should be noted that in the experience of everyday life, including that of the engineer's workshop and the scientific laboratory, we have to do for the most part with motions of small duration and extent. Also that we always refer such motions to points of the earth's surface at the places of observation. The Second Law as based on such experience should therefore be enunciated as holding for motions of the above restricted kind, and provided they are specified in the way mentioned.

Astronomers tell us that in their study of the motions of the heavenly bodies, they have found the same law to hold, but only provided motions are specified relatively to points fixed in the celestial sphere.

The question therefore arises: Are there two laws, or is one a particular case of the other? The question is perhaps too difficult for discussion at our present stage. Nor is it necessary to discuss it. For it has been practically settled by the application of both laws to the discussion of cases of motion on the earth's surface which are of considerable duration or extent, *e.g.*, the oscillation of pendulums for long times, the behaviour of gyrostats,

the falling of bodies through great distances, etc., and it has been found that the results predicted by the astronomical law agree with observed fact, while those predicted by the law given by ordinary experience depart more widely therefrom the greater the duration or the greater the extent of the motion considered.

The Second Law is therefore to be regarded as holding generally relatively to points of reference fixed in the celestial sphere, and as holding relatively to points fixed in the earth's surface at any place at which a motion is being observed, only provided the motion is of sufficiently small duration and extent. In doubtful cases the astronomer's law must be used, or as it is usually expressed, the effect of the rotation of the earth must be allowed for.

292. By 117, the acceleration of a body is equal to the rate of change of its component velocity in the direction of the acceleration; and, by the second part of the above experiential result (287), the acceleration is in the direction of the force. Hence, if v and v' be the initial and final values of the body's component velocity in the direction of the force (F) during a time t (which, if F is variable, must be small), we have

$$F = km\frac{v'-v}{t} = k\frac{mv'-mv}{t}.$$

293. The product of the mass of a body into its velocity is called its *momentum*.* The product of its mass into the component of its velocity in a given direction is called its component momentum in that direction.

Hence the result of 292 may be thus expressed: When a body is acted upon by a force its component momentum in the direction of the force changes at a rate which is proportional to the force.

* The momentum of a body is often defined as its "quantity of motion," the quantity of motion being then defined as the product of mass into velocity—another case of the introduction of a useless intermediate term.

294. From the expression of 292 we obtain

$$Ft = k(mv' - mv).$$

The product Ft is called the *impulse* of the force during the time t. Hence we obtain Newton's expression of the second law of motion—

Change of momentum is proportional to the impulse of the impressed force and takes place in its direction.

295. It follows from the second law of motion that a finite force can produce in a body only a finite change of momentum and therefore a finite acceleration. As we find no infinite forces in nature it follows that the speeds, velocities, and accelerations of bodies cannot have infinite values, and that the directions of their paths, velocities, accelerations cannot undergo abrupt changes.

296. *Measurement of Force and Mass.*—The second law gives us at once a mode of measuring both force and mass. If forces F, F' act on bodies of mass m and m' and produce accelerations a and a' respectively, we have $F = kma$ and $F' = km'a'$. To compare two forces, allow them to act successively on the same body and note the accelerations. We have then

$$F : F' = a : a'.$$

To compare the masses of two bodies, let equal forces act on them and note the accelerations. We then have

$$m : m' = a' : a.$$

297. *Units of Force and Mass.*—Having thus obtained modes of measurement, we must next select standards or units. Both the unit of force and that of mass may be selected arbitrarily, or one having been so selected, the other may be derived.

If both are selected arbitrarily the constant k, for the particular set of units, in the equation of the Second Law,

$F=kma$, must be determined by experiment. Thus if the force exerted by a certain spring when stretched a certain amount be selected as the standard force and the mass of a certain body as the standard mass, we may allow the standard force to act on the standard body, and note the acceleration produced. If it has, say, the value 24·1, we find, by substituting the values $F=1, m=1$, and $a=24{\cdot}1$ in the above equation, that $k=1/24{\cdot}1$. We must use this value of k in the equation of the Second Law when we wish to employ the above units of force, mass, length and time.

298. *Gravitational Units.*—The most important systems of arbitrarily selected units are those in which the weight of a standard body is adopted as unit of force, and its mass as unit of mass. Such systems are consequently called Gravitational systems. They are in almost universal use in most forms of engineering work and are thus frequently referred to as Engineers' units.

As the weight of a body is different at different places on the earth's surface, the units of force of such systems are not everywhere the same; but their variation from place to place is so slight as to occasion no practical inconvenience.

As the weight of a body is known from experiment to produce in the body, in falling freely, an acceleration whose value depends only on the units of length and time employed in its specification, and is usually indicated by g, the constant k in the equation $F=kma$, will for such a system have the value $1/g$.

Any body may be selected as the standard body; but the more important of those which are actually employed are the *pound*, which is a piece of platinum preserved in the Standards Office in London, and the *kilogramme*, another piece of platinum preserved in the Palais des Archives in Paris, and multiples or submultiples of these.

299. Corresponding to each standard body selected and to the standards of length and time that may be used, we have a system of gravitational units. The more important are as follows:—

The Foot-pound-second (F.P.S.) Gravitational System.—The unit of force is the weight, and the unit of mass, the mass, of the pound. The equation of the Second Law takes the form

$$F = \frac{1}{32 \cdot 2} ma.$$

The Metre-kilogramme-second (M.K.S.) Gravitational System.—The unit of force is the weight, and the unit of mass, the mass, of the kilogramme. The equation takes the form

$$F = \frac{1}{9 \cdot 81} ma.$$

In both cases the constant varies slightly from place to place.

Other gravitational systems based on other standard bodies, the reader will readily construct for himself. The above are sufficient for purposes of illustration. The value of the constant is determined obviously, not by the standard body but by the units that may be employed for length and time.

When the units of force and mass are determined as above by the selection of a standard body, the constant in the equation expressing the Second Law will always have the value $1/g$. If instead of employing as unit of mass the mass of the standard body, we employ g times its mass, the constant k will take the convenient value unity. For a force equal to the weight of a body will produce in a body of g times as great mass an acceleration unity. Hence if we agree to employ g times the mass of the standard body as unit we may write the Second Law, $F = ma$.

300. *Derived or Absolute Units.*—The constant k may have the convenient value unity given it by selecting arbitrarily only one of the two units of force and mass, and deriving the other in an appropriate manner. For as the value of k depends only upon the units employed in the above equation, it can be given any value we choose, even if three of the units have already been selected, by a proper selection of the fourth. A system of units in which either the unit of force or that of mass is selected with this object is called a derived system.

Practically it is always the unit of mass that is arbitrarily selected and the unit of force which is derived, the reason being that it is so much easier to preserve a standard body than to preserve an instrument which will exert a standard force.

The units ordinarily selected are the mass of the pound and that of the gramme (a body whose mass is 1/1000th of that of the kilogramme), with their multiples and submultiples. The English hundredweight is equal to 112 pounds, the American hundredweight to 100 pounds. The ton is equal to 20 cwts. The decagramme and hectogramme are 10 and 100 grammes respectively. The decigramme, centigramme, and milligramme are the tenth, hundredth, and thousandth parts respectively of a gramme.

The following are approximately the relative magnitudes of these units:

1 lb. = 453·59 grm.	1 grm. = 0·0022046 lb.
1 ton (English) = 1016·05 kgr.	1 kgr. = 0·0009842 ton.

301. In the equation $F=kma$, k will be unity, if $F=1$, $m=1$, and $a=1$, if therefore the force which produces an acceleration equal to unity in a body of mass unity, is expressed in terms of such a unit as to have itself the value unity. Units of force which would make $k=1$ might be derived otherwise; but the above method

is the simplest. The derived unit of force in any system is thus that force which will produce acceleration unity in a body of mass unity.

As the mass of a body is constant, and the magnitudes of derived units depend only on those of the simple units involved in them, units of force derived as above are constant. To indicate their independence of gravity, they are frequently called absolute units.

As pointed out above, the constant k will become unity if forces and masses are expressed in terms of gravitational units provided g times the mass of the standard body be selected as unit of mass. With this proviso, then, the gravitational unit of force may be regarded as derived from this unit of mass after the same manner as the unit of force of an absolute system.

302. Corresponding to each unit of mass selected, we have a system of absolute units. The following are important systems:—

Foot-pound-second (*F.P.S.*) *Absolute System.*—The unit of mass is the mass of the pound.

The unit of force is therefore that force which will produce in the pound an acceleration of 1 ft.-sec. unit. This force is called the *poundal.* As the weight of 1 lb. produces in it an acceleration of 32·2 ft.-sec. units, it is clear that the poundal is equal to the 1/32·2th part of the weight of a pound, *i.e.*, to about the weight of half an ounce.

Centimetre-gramme-second (*C.G.S.*) *Absolute System.*—The unit of mass is the mass of the gramme.

The unit of force is therefore that force which will produce in 1 gramme an acceleration of 1 cm.-per-sec. per sec. This force is called the *dyne.* It will be clear that the dyne is equal to about 1/981th of the weight of a gramme.

303. *Dimensions of Derived Unit of Force.*—The magnitude of a unit of force, derived as above, will depend upon the magnitudes of the simple units of mass, length, and time involved in it. With the notation of 15, we have (289 and 15)

$$F=ma\,;\quad F\propto\frac{1}{[F]};\quad m\propto\frac{1}{[M]};\quad a\propto\frac{1}{[a]}.$$

Hence $[F]\propto[M][a]$, *i.e.*, the magnitude of the derived unit of force is directly proportional both to the magnitude of the unit of mass and to that of the unit of acceleration. Hence (111 and 57) the dimensions of the derived unit of force are given by the equation

$$[F]\propto[M][L][T]^{-2}.$$

This equation may be employed in the solution of problems in the same way as the corresponding equations in the case of speed and rate of change of speed (47–50, 57–59).

304. *Density.*—The *mean density* of a body is the quotient of its mass by its volume.

The *density at a given point* of a body is the quotient of the mass by the volume of an indefinitely small portion of the body surrounding the given point. If the density of a body is the same at all its points, it is said to be *homogeneous* or of uniform density. In general the density of a body varies from point to point; the body is heterogeneous.

The density of a substance in a given state is the quotient of the mass by the volume of any portion of the substance in that state.

If d be the density of a body, m its mass, and v its volume, we have by definition $d=m/v$. Hence the dimensions of density are given by the expression

$$[D]\propto[M][V]^{-1}\propto[M][L]^{-3}$$

The unit of mass of an absolute system of units, instead of being arbitrarily selected as in 301, may be defined to be the mass of unit volume of some standard substance whose density in terms of those units is therefore unity. This amounts to choosing a unit of density arbitrarily and deriving from it the unit of mass. The French unit, the gramme, was intended to be the mass of 1 cubic centimetre of water at its temperature of maximum density (about 4° C.). But though it may for most practical purposes be considered to have that mass, it has not rigorously; and thus the gramme must be considered to be an arbitrarily chosen unit. The great advantage of deriving the unit of mass from an arbitrarily chosen unit of density is that the density of any given substance is in that case equal to the ratio of the masses (and therefore (290) of the weights) of equal volumes of the given substance and of the standard substance, or to what is called the *specific gravity* of the given substance. If the unit of mass is not thus derived, the density of a given substance is obviously equal to the product of its specific gravity into the density of the standard substance (usually water at 4° C.) by reference to which its specific gravity is expressed.

The *mean linear density* of a body whose length is great relatively to its other dimensions is the quotient of its mass by its length. The dimensions of linear density are thus $[M][L]^{-1}$.

The *mean surface density* of a thin body is the quotient of its mass by the area of one of its surfaces. The dimensions of surface density are thus $[M][L]^{-2}$.

305. *Examples.*

(1) Two forces produce in two bodies accelerations of 25 and 30 units respectively. Show that, if the masses are equal, the forces are as 5 to 6, and that, if the forces are equal, the masses are as 6 to 5.

(2) Forces of 20 and 30 units acting on two bodies produce accelerations of 40 and 50 units respectively. Show that the masses are as 10 : 12.

(3) Show that 1 poundal is equivalent to 13,825 dynes.

(4) Prove that the weight of 1 lb. is equal to $4{\cdot}45 \times 10^5$ dynes approximately.

(5) Show that the value of 1 dyne, expressed in terms of the weight of 1 ton,* is 1003×10^{-12} approximately.

(6) Compare the values of the mass of a body as expressed in gravitational units of the ft.-lb.-sec. and yd.-ton-min. systems.

Ans. 2,688,000 : 1.

(7) The value of a force expressed in dynes has to be expressed in absolute units of the metre-kilogramme-minute system. By what number must it be multiplied?

Ans. 0·036.

(8) Reduce 20 poundals to absolute units of the yd.-cwt.-min. system.

Ans. $214\frac{2}{7}$.

(9) The unit of mass being a mass of 10 lbs., the unit of time 1 min. and the unit of length 1 yd., compare the derived unit of force with the poundal.

Ans. As 1 : 120.

(10) With 20 lbs. and 40 sec. as units of mass and time respectively, find the unit of length that the derived unit of force may be equal to the weight of 1 lb. at a place where $g = 32{\cdot}2$ ft.-sec. units.

Ans. 2,576 ft.

(11) The unit of acceleration being 6 ft.-per-sec. per sec., find (*a*) the unit of mass when the derived unit of force is equal to the weight of 20 lbs., and (*b*) the unit of force when the derived unit of mass is a mass of 20 lbs.

Ans. (*a*) $107\frac{1}{3}$ lbs., (*b*) 3·7... pounds-weight.

(12) The unit of velocity being 20 cm. per sec., the unit of mass 15 grammes, and the derived unit of force the weight of a kilogramme, find the unit of time.

Ans. 1/3270 sec.

* The ton used in these Examples is the English ton of 2,240 lbs.

(13) The density of water is about 1,000 oz. per cub. ft. Show that it is also about 1687·5 lbs. per cub. yd., and about 1·001 grm. per cub. cm.

(14) The masses and radii of two spheres are as 1 : 2. Show that their densities are as 4 : 1.

(15) Given that the diameter of the earth is $1{\cdot}275 \times 10^9$ cm. and its density 5·67 times as great as that of water, show that its mass is about $6{\cdot}15 \times 10^{27}$ grammes.

(16) The unit of density being that of water, and the units of time and mass 1 min. and 1 cwt. respectively, find the magnitude of the derived unit of force.

Ans. 0·0378 poundals nearly.

(17) The number of seconds in the unit of time being equal to the number of feet in the unit of length, the unit of force being the weight of 750 lbs. ($g=32$ ft.-sec.-units), and a cub. ft. of the standard substance having a mass of 13,500 oz., find the unit of time.

Ans. $5\frac{1}{3}$ sec.

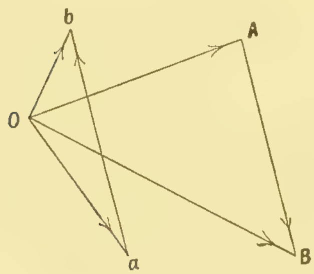
306. Force is usually exerted upon some portion of the bounding surface of a body and acts therefore across an area. In specifying the magnitude of a force we may do so, as above, without reference to the area across which it acts, or we may divide its total magnitude by this area and thus express its magnitude per unit of area or its intensity. When we do so we usually describe the force as a pressure, a tension, a stress, though these terms have another not inconsistent connotation (307) as well. Thus a force of F poundals which is transmitted by a string of s square feet section would be a tension of F/s poundals per square foot, and the total force exerted through the string would of course be determined by multiplying this quantity by the area across which the force is acting.

It should be noted however that forces are not always measured in this way when they are spoken of as pressures, tensions, etc. Unless either it is stated, or the context shows, that they are so measured, they should

always be assumed to be measured without reference to the area across which they act.

307. *Third Law of Motion.*—If we now return to the examination of cases in which bodies are acted upon by forces, we find that forces always act between pairs of bodies, never on single bodies alone. I push a body with my hand; the body is urged forwards; the forward motion of my hand is lessened. Both the body pushed and the hand are acted upon by force. A horse draws a carriage; the carriage is pulled forwards; the horse is pulled backwards and does not move forwards so fast as he would otherwise do with the muscular exertion he is putting forth.

If we wished to investigate this mutual action experimentally we might project two of our curling stones, on the ice, without rotation, so as to make them collide, noting the direction and magnitude of their velocities before and after collision. Let OA, OB and Oa, Ob be drawn representing the velocities before and after collision, of the respective stones. Then AB, ab will represent the respective integral accelerations. They would be found in all cases, so far as such experiments could show, to be parallel and in opposite directions.



If the stones used were of equal mass, they would be found equal. If not, it would be found that if M, m were the masses of the respective stones, then $M\,.\,AB = m\,.\,ab$. Now the product of the mass of a body (294 and 117) into its integral acceleration measures the impulse of a force. Hence the stones would be found to have experienced equal impulses in opposite directions during the collision.

Although few of us have performed such experiments as these, we have all recognized in our own experience

more or less clearly the equality and opposition of the forces acting between two bodies, at least between bodies which are not in motion. And our confidence in this result, as in the case of the second law, is really based in part on our own experience, and in part on the fact that the application of the law in engineering practice and in astronomical calculations has produced results which are borne out by fact.

Newton enunciated the law of the mutual action of bodies as follows:—

To every action there is always an equal and contrary reaction; or the mutual actions of any two bodies are always equal and oppositely directed.

Some persons find it difficult to admit the equality of action and reaction in cases in which the reacting bodies are in motion. If the horse is pulled back by the carriage, they say, with a force equal to that with which he pulls the carriage forward, how is it that motion ensues at all? If the second law be admitted, however, the acceleration of the carriage is seen to depend only on the forces acting on it, and not on forces which may be acting on the horse. Whatever may be the forces by which the horse is pulled back, the carriage will have a forward acceleration if the forward forces acting on it exceed those in the reverse direction. So also in the case of the horse. Whether or not he is to have a forward acceleration depends entirely on the relative magnitude of the forward and backward forces acting on him. If through muscular exertion he is able to push the earth backwards through his hoofs, and thus cause the earth to push him forwards, with a force which is greater than the backward pull exerted by the carriage, he will have a forward acceleration. And thus the possibility of motion ensuing in such a case is independent of the relative magnitude of the action and reaction involved.

The exertion of a force upon one body is thus only a

one-sided view of a more complex phenomenon, viz., the simultaneous exertion of equal and opposite forces upon two bodies. When we are thinking of a force as acting not on one body, but between two bodies, we call it a stress. When the stress is such as to make the bodies move towards one another, it is called an attraction or a tension; when its effect is to increase their distance it is called a repulsion or a pressure (see 306).

308. The laws of motion may perhaps be regarded as direct generalizations from experience, so far as their application to the translation of finite bodies is concerned. But in the study of rotation we are forced to regard bodies as made up of indefinitely small parts, and the laws of motion are applied to them. This application obviously transcends experience. Hence the laws of motion as employed in Dynamics are simply hypotheses suggested by experience, and their accuracy must be tested by the agreement of deductions made from them with observed fact. The body of deductions from these hypotheses constitutes the theoretical portion of Dynamics. Many of the deductions which will be made in subsequent chapters may be tested by experiment. But for the most part we shall have to do with ideal bodies and our deductions will be only approximately true of real bodies. The most satisfactory tests of the laws of motion are furnished by astronomical calculations. These laws are assumed in the determination of the positions of the moon and other heavenly bodies at given times, of the dates of the return of comets, etc., and the precision with which such predictions are fulfilled is well known. The assumption of them has even led to the discovery of heavenly bodies not previously known to exist. They have thus stood the most rigorous tests. And we make deductions from them, even in cases in which verification by experiment is impossible, with full confidence that, if our mode of deduction is correct, the result will be consistent with fact.

309. The three laws of motion adopted by Newton as the fundamental hypotheses of Theoretical Dynamics have not been universally adopted. Some authors substitute for Newton's second law one first enunciated by Galileo, and therefore bearing his name, which has been expressed by Thomson and Tait in the following words:—

When any forces whatever act on a body, then whether the body be originally at rest or moving with any velocity in any direction, each force produces in the body the exact change of motion which it would have produced if it had acted singly on the body originally at rest.

As Newton's second law is perfectly general it includes Galileo's law. Those who make Galileo's law the second law of motion must deduce Newton's law from it. This deduction is made as follows:—Let two forces each equal to F act in the same direction on a particle. Then if a is the acceleration which each would produce if it acted singly, $2a$ is by Galileo's law the acceleration produced when they act together. Similarly $3a$ is that which would be produced by three forces each of the magnitude F and in the same direction; na that which would be produced by n such forces. And hence the acceleration produced in the particle is proportional to the force. It will be noticed that the assumption is here made that n equal forces in the same direction are equivalent to a force of n times the magnitude, a special case of the Law of the Composition of Forces (313 and 86, iii). We made the same assumption in discussing the rough experiments used to suggest the fundamental hypotheses. But such an assumption made after the choice of three fundamental hypotheses is equivalent to the introduction of a fourth.

For D'Alembert's Principle, which is extensively employed instead of Newton's second and third laws in the solution of problems on the motion of extended bodies, see 417.

For "the impossibility of the perpetual motion" as a law of motion, see 436.

CHAPTER II.

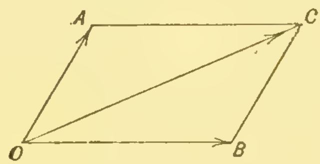
DYNAMICS OF A PARTICLE.

310. We shall first restrict ourselves to the consideration of force as affecting the translation of bodies. Now the translation of an indefinitely small body differs in no respect from that of a body of finite size, while rotation is possible only for bodies of finite size. Hence in considering the effect of force on the translation of bodies, in order to exclude the possibility of its having rotational effects, we imagine the bodies acted upon to be indefinitely small. Such bodies are called material points or particles, or, if they form parts of a continuous body, elements.

311. A force which we imagine as acting on a particle is of course one whose place of application is a point. The lines of action of forces which act on the same particle must intersect in the position of the particle. A force is completely specified if its place of application, its direction, and its magnitude are given. If it act on a particle, its place of application is the position of the particle itself. In that case therefore it is completely specified if its direction and magnitude are given. It may therefore be completely represented by any straight line of the proper length and direction.

312. *Composition and Resolution of Forces.*—Forces which act simultaneously on a particle are called component forces. The resultant of any number of component forces is that single force by which the same resultant acceleration would be produced.

313. Let OA and OB represent two component forces. Since these forces act upon the same particle, OA and OB represent also the accelerations they would produce acting singly. Now OA and OB representing the component accelerations, OC the diagonal of the parallelogram AB represents the resultant acceleration (116). And OA, OB, OC representing accelerations of the same particle are proportional to the forces which would produce them. Hence OC represents the resultant force.



Forces acting on a particle therefore are to be compounded according to the parallelogram law after the manner of the displacements, or velocities, or accelerations of a point. We have therefore propositions called the parallelogram, the triangle, and the polygon of forces, the same in form as those enunciated under velocities (98). Hence forces are to be resolved in the same manner as displacements, or velocities, or accelerations.

Hence all the consequences of the parallelogram law, as deduced in the case of the displacements of a point, apply also to forces acting on a particle, and the formulae of 85–90 are applicable to component forces, if the symbols representing displacements are taken to represent forces.

314. *Examples.*

(1) The resultant of forces of 7, 1, 1, 3 units represented in direction by lines drawn from one angle of a regular pentagon towards the other angles, taken in order, is $\sqrt{71}$.

(2) P and Q are two component forces whose resultant is R. S is the resultant of R and P. Show that if P and Q be inclined to each other at a right angle, and if $Q=2P$, then $S=2P\sqrt{2}$.

(3) Component forces P, Q, R are represented in the direction by the sides of an equilateral triangle taken the same way round. Find the magnitude of their resultant.

Ans. $(P^2+Q^2+R^2-QR-PR-PQ)^{\frac{1}{2}}$.

(4) Three component forces are represented by lines drawn from the angular points of a triangle to the points of bisection of the opposite sides; show that their resultant is zero.

(5) Three component forces are represented in direction by lines drawn from the angular points A, B, C of a triangle to the points of bisection of the opposite sides, and have magnitudes equal to the cosines of A, B, and C respectively. Prove that their resultant is equal to $(1-8\cos A\cos B\cos C)^{\frac{1}{2}}$.

(6) The centre of the circumscribed circle of a triangle ABC is O, and the intersection of the perpendiculars from angular points on opposite sides is P. Prove that the resultant of forces represented in magnitude and direction by OA, OB, OC will be represented by OP.

(7) Three component forces are represented by the sides AB, AC, BC of a triangle. Show that the resultant has the direction AC and is represented in magnitude by $2AC$.

(8) $ABCD$ is a parallelogram. From AB, AE is cut off equal to one-third of AB. Prove that the resultant of forces represented by AC and $2AD$ is equal to three times the resultant of forces represented by AD and AE.

(9) If AB represent the resultant of two forces AC and AD, and if the angle CAD be given, show that the extremities of the lines representing the two forces (AC and AD) will lie on two circles, which, if the given angle be a right angle, will be coincident. Also show that, if the given angle be obtuse, each force has its maximum value when the other is perpendicular to the resultant.

(10) A particle is acted upon by two forces represented by the lines joining the particle to two given points. Show that, if the

particle be made to describe any plane curve, the end of the straight line representing the resultant of the above forces will describe an equal and similar curve.

(11) Give a geometrical construction for resolving the force represented by the diagonal DB of a square $ABCD$ into three forces, each represented in magnitude by a side of the square and one represented by DC in direction.

Ans. Upon BC describe an equilateral triangle BCE. The required components are represented by DC, CE, EB.

315. *Attractions.*—An important case of the composition of forces is the determination of the attractive force exerted on a particle by an extended body, the law of the attraction being that of gravitation, viz., that the force exerted between two particles is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In such cases the attraction on the particle is the resultant of component attractions exerted on it by the elements into which the attracting body may be divided. Its determination requires usually the application of the Integral Calculus. But in a few important cases it may be found by elementary methods.

If m, m' are the masses of two particles, d their distance, and F their mutual attraction, the law of gravitational attraction is expressed by the equation

$$F \propto \frac{mm'}{d^2} = k\frac{mm'}{d^2},$$

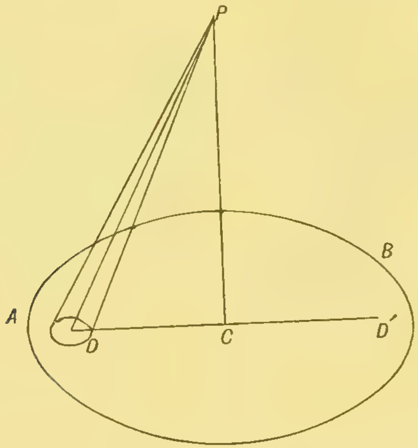
where k is a constant. The value of k, when units of force, mass, and distance, already chosen, are employed, may readily be determined from our knowledge of the dimensions and density of the earth and of the value of g (316, Ex. 13). We may give it the more convenient value unity, however, by choosing a new unit of mass, taking as unit of mass the mass of a body which attracts a body of equal mass at unit distance with

unit force. This is called *the astronomical unit of mass* (316, Ex. 12). We shall use it in the following examples.

316. *Examples.*

(1) Find the attraction of a uniform thin circular disc on a particle placed at any point on a line through its centre and perpendicular to its plane.

Let AB be the disc, C its centre, CP a line through C perpen-



dicular to AB. Let P be the position of the particle, and m its mass.

Consider first the component attraction exerted by the element (*i.e.*, small portion) of the disc surrounding any point D, the line DP having the length r, and its inclination to CP being θ radians. Let the element at D subtend at P the small solid angle ω (solid radians, 22). DP's inclination to CP being θ, the surface of the element at D is inclined to a surface normal to DP at the same angle. The element at D being indefinitely small, the cone of which it is a section is one of indefinitely small angle. Hence the orthogonal section of this cone at D is the projection of the element at D on a plane inclined θ to the plane of the element. If therefore A is the area of the element, $A \cos \theta$ is the area of the orthogonal section. But ω being the solid angle subtended at P by this section, its area must be ωr^2. Hence the area of the element at D

is $\omega r^2/\cos\theta$. Let ρ be the surface density of the disc. Then the mass of the element is $\omega r^2\rho/\cos\theta$. Hence the force exerted by the element on the particle at P is in the direction of PD and of the magnitude

$$\frac{\dfrac{\omega r^2\rho}{\cos\theta}\times m}{r^2}=\frac{\omega\rho}{\cos\theta}\cdot m.$$

This force has one component in the direction PC of the magnitude

$$\frac{\omega\rho}{\cos\theta}m\times\cos\theta=\omega\rho m,$$

and another in the direction CD of the magnitude $\omega\rho m\tan\theta$.

If DC be produced to D', and CD' made equal to CD, the element of area A at D' will exert on the particle at P a force whose components in the directions PC, CD' are of the same magnitudes as the components above determined. The components CD and CD' therefore neutralize each other, and hence the only effective component of the attraction of the element at D is that perpendicular to the disc, whose magnitude is $\omega\rho m$.

Now the same is true of all the elements into which the disc may be divided. Hence the resultant attraction will be perpendicular to the disc, and equal to the sum of the effective components of magnitude $\omega\rho m$, for all the elements of the disc, *i.e.*, since ρm is constant, to the product of ρm into the solid angle subtended at P by the whole disc. If a is the radius of the disc and h the distance of P from it, the area of the segment, bounded by AB, of the sphere whose centre is P and radius PA or $\sqrt{h^2+a^2}$ is $2\pi\sqrt{h^2+a^2}(\sqrt{h^2+a^2}-h)$. Hence the solid angle subtended at P by the disc is $2\pi(1-h/\sqrt{h^2+a^2})$; and therefore the attraction of the disc on the particle at P is $2\pi\rho m(1-h/\sqrt{h^2+a^2})$.

If the disc be of indefinitely great extent ($a=\infty$), or if the particle be indefinitely near it ($h=0$), the attraction becomes $2\pi\rho m$.

(2) Find the attraction of a thin circular ring of gravitating matter of uniform linear density ρ and radius a on a particle of unit mass on its axis, at a distance h from its centre.

Ans. $2\pi\rho ah/(a^2+h^2)^{\frac{3}{2}}$.

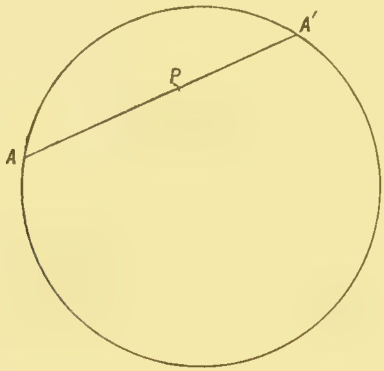
(3) All parallel slices of equal thickness of a homogeneous cone of gravitating matter exert the same attraction on a particle at its

vertex. [First prove for a cone of indefinitely small angle and then extend to one of finite angle.]

(4) A right cone of gravitating matter of semi-vertical angle α, length l, and uniform density ρ, attracts a particle of unit mass at its vertex with a force $2\pi\rho l(1-\cos\alpha)$.

(5) Show that the attraction of a thin spherical shell of uniform thickness and density on a particle inside it is zero.

Let P be the position of the particle, and A any point in the spherical surface. Join AP and produce it to meet the surface in A. Consider a small element of the shell at A. If, from points

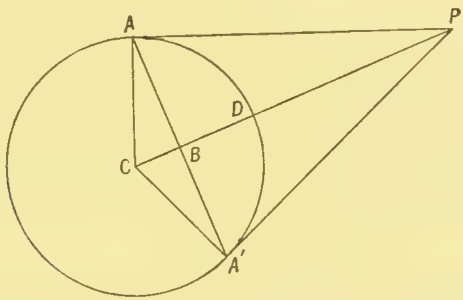


in its boundary, lines be drawn through P, their end points will mark off a corresponding element about A'. These corresponding elements are both sections of the same cone, and as they coincide with the tangent planes at A and A', we know from the geometry of the sphere that they are equally inclined to the line AA'. Hence their areas, and therefore their masses, are directly proportional to the squares of their distances from the vertex P. But their attractions on a particle at P are directly proportional to their masses and inversely proportional to the squares of their distances from P. Hence their attractions have the same magnitude. And they have opposite directions. Hence the pair of elements about A and A' exert no resultant attraction on the particle at P. But the whole shell may be divided into such pairs of elements. Hence the resultant attraction of the shell on a particle at P is zero.

Clearly the same would hold for a shell of any thickness, provided it is either uniform in thickness and density, or uniform in thickness and symmetrical about the centre as to density.

(6) The attraction of a uniform thin spherical shell on a particle placed outside it is the same as if the whole mass were condensed at the centre.

Let P be the position of the particle and C the centre of the spherical shell. Join CP, meeting the shell in D, and divide it



at B, so that $CB : CD = CD : CP$. Take any point A in the shell. Join AB and produce it to meet the shell in A'. Join CA, CA', PA, PA'.* Since $CB : CA = CA : CP$, the triangles CAB and CPA are similar, and the angle CAB equal to the angle CPA. Similarly, the angle $CA'B$ is equal to the angle CPA'. Hence also the angle CPA is equal to the angle CPA'.

If straight lines be drawn from the boundary of a small element surrounding A, through B, their end points will mark out a corresponding element about A'. These elements are sections of a cone whose vertex is B and solid angle ω (solid radians); and their common inclination to an orthogonal section of the cone, is the angle CAB. Hence, as in Ex. (1), their attractions on a particle of mass m at P are respectively in the directions PA and PA', and of the magnitudes

$$\frac{m\rho\omega \, . \, AB^2}{AP^2 \, . \cos CAB} \text{ and } \frac{m\rho\omega \, . \, A'B^2}{A'P^2 \, . \cos CAB},$$

ρ being the surface density of the shell. Now

$$\frac{AB}{AP} = \frac{CA}{CP} = \frac{CA'}{CP} = \frac{A'B}{A'P}.$$

* PA and PA' are not tangents, as would appear from the figure.

Hence the magnitudes of the above attractions are equal, and they are equally inclined to PC. Hence the direction of their resultant is PC, and its magnitude is $2m\rho\omega \,.\, CA^2/CP^2$. Now the whole sphere may be divided by lines through B into pairs of corresponding elements similar to the above, the resultant attraction of each pair being in the direction PC, and equal to the product of its solid angle into the constant $2m\rho \,.\, CA^2/CP^2$. Hence the resultant attraction of the spherical shell is in the direction PC, and is equal to the product of this constant into the sum of the solid angles of all the pairs of elements into which the sphere may be divided, which is clearly the solid angle subtended at its centre by a hemisphere. Its magnitude is therefore $4\pi m\rho \,.\, CA^2/CP^2$, which is equal to the product of the masses of the particle and shell divided by the square of the distance of the particle from the centre of the shell. Hence the shell attracts the particle as if its mass were condensed at its centre.

Hence also a spherical shell of any thickness, and a sphere also, attract particles outside them as if their masses were condensed at their centres, provided their density is symmetrically distributed about their centres.

(7) Show that the attraction of a homogeneous sphere on a particle of unit mass inside its bounding surface is directly proportional to its distance from the centre.

(8) Assuming the earth to be a homogeneous sphere, compare its attraction on a given body at a distance from its centre equal to one-half its radius with the attraction when the given body is at a distance equal to twice the radius.

Ans. As 1 : 8.

(9) Find in dynes the attraction of two homogeneous spheres, each of 100 kgr. mass, with their centres 1 metre apart. [Data.—Quadrant of earth, assumed spherical $=10^9$ cm.; mean density of earth $=5{\cdot}67$ grms. per cu. cm.; $g=981$ cm.-sec. units.]

Ans. 0·0649 nearly.

(10) A pendulum beating seconds at the surface of the earth is taken (a) up a mountain 1,400 ft. high, and (b) down a mine of

equal depth. Find its loss or gain per day in each case, assuming the earth to be a uniform sphere of 21,000,000 ft. radius.

Ans. (*a*) loss of 5·76 sec.; (*b*) loss of 2·88 sec.

(11) Show that, if a pendulum oscillates in the same time at the top of a hill as at the bottom of a mine, the depth of the mine is nearly twice the height of the hill.

(12) Show that the astronomical unit of mass of the C.G.S. system is 3,928 grammes (mass of earth $= 6{\cdot}14 \times 10^{27}$ grms.; radius of earth $= 6{\cdot}37 \times 10^{8}$ cm.; $g = 981$ cm.-sec. units).

(13) Find in C.G.S. units the value of k in the formula $F = k\dfrac{mm'}{d^2}$.

Ans. $6{\cdot}48 \times 10^{-8}$.

(14) Compare with the dyne the unit of force employed when it is stated that the attraction between two bodies of m and m' grms. at a distance d cm. has the value mm'/d^2.

Ans. Unit employed $= 6{\cdot}48 \times 10^{-8}$ dynes.

317. *Equations of Motion.*—The second law of motion provides us with an equation, $F = ma$, by means of which any one of the three quantities, force acting, mass of particle acted upon, and acceleration produced, may be determined, if the other two are given. These two being expressed in the units of a derived system, the third determined by the above equation will be expressed in terms of the unit of the same system.

The acceleration of a particle being determined, the character of its motion is known from Kinematics. Hence the above equation is called the equation of motion of a particle.

If a particle is given as acted upon by several forces, the resultant force may be found as in 313, or, the component accelerations having been found by the equation of motion, the resultant acceleration may be determined by 116.

It follows that if F_1, F_2, etc., are the components in a

given direction of the forces acting on a particle, and a its component acceleration in that direction, $\Sigma F = ma$.

318. It is in many cases found convenient in describing the forces acting on particles, to specify not their magnitudes and directions, but the magnitudes of their components in three given rectangular directions. Expressions for the component accelerations in these three directions may be at once written down. Thus, if X_1, X_2, etc., Y_1, Y_2, etc., Z_1, Z_2, etc., be the components in the x, y, z axes of the forces acting on the particle, if a_x, a_y, a_z be the component accelerations of the particle in these directions, and x, y, z the co-ordinates of the particle at the instant under consideration, we have (317 and 118)

$$a_x = \ddot{x} = (\Sigma X)/m,$$
$$a_y = \ddot{y} = (\Sigma Y)/m,$$
$$a_z = \ddot{z} = (\Sigma Z)/m.$$

319. It is frequently convenient to express the equation of motion in terms of the impulse of the force rather than of the force itself. If the force (F) is constant, its impulse (Φ) during a time t is (294) Ft, and we have from 294 and 298

$$\Phi = Ft = mv' - mv,$$

where v' and v are the final and initial values of the component velocity in the direction of the impulse, and m is the mass of the particle acted upon.

If the force is variable, it may be considered constant during indefinitely short intervals of time. Let t be divided into n such short intervals t_1, t_2, etc., t_n. Let the components, in any given direction, of the force (supposed constant) during these intervals be F_1, F_2, etc., F_n; and let the components, in the given direction, of the initial velocity and of the velocities at the ends of the above intervals be v, v_1, v_2, etc., v' respectively. Then

$$F_1t_1 = mv_1 - mv,$$
$$F_2t_2 = mv_2 - mv_1,$$
$$\text{etc.},$$
$$F_nt_n = mv' - mv_{n-1}.$$

The impulse (Φ) of the force, in the given direction, is the sum of the impulses F_1t_1, F_2t_2, etc. Hence

$$\Phi = \Sigma Ft = mv' - mv.$$

This form of the equation of motion is especially convenient when the force is one whose magnitude is great and time of action small, as in cases of impact, collision, explosion, etc. Such forces are therefore frequently called *impulsive forces.* It will be obvious however that the above form of the equation of motion is applicable generally, and that the restriction of the term impulsive force to one whose time of action is short is merely a matter of convenience.*

320. *Examples.*

(1) A constant force of 20 poundals acts on a body of 10 lbs. Find (a) the acceleration, (b) the displacement in 5 sec., the initial velocity having been 4 ft. per sec. in the same direction as the acceleration; (c) the velocity at the end of the same time, the initial velocity of 4 ft. per sec. having been inclined 60° to the direction of the force.

Ans. (a) 2 ft.-per-sec. per sec., (b) 45 ft. in the direction of the initial velocity, (c) $2\sqrt{39}$ ft. per sec., inclined $\sin^{-1}(5/2\sqrt{13})$ to the direction of the initial velocity.

* The term impulse is unfortunately sometimes applied to these short-lived forces. But it should be restricted to the sense in which it is used above. Otherwise it becomes necessary to speak of the impulse of an impulse. The term impulsive force is sometimes used to denote the impulse of a short-lived force. But this use of the term leads to confusion and should be avoided.

(2) An unknown force produces in a body of 50 lbs. mass an acceleration of 12·5 ft.-sec. units. Express the force (a) in poundals, (b) in terms of the weight of a pound.

Ans. (a) 625, (b) 19·4....

(3) A uniform force of 200 dynes changes the velocity of a body moving in a straight line from 250 to 300 metres per sec. in 1 minute. Find the mass of the body.

Ans. 2·4 grammes.

(4) What acceleration will be produced in a body weighing 20 lbs. by a force equal to the weight of 50 lbs.?

Ans. $5g/2$.

(5) How long must a force of 14 lbs.-weight act on a mass of 1,000 tons to move it from rest through 1 inch?

Ans. 28·8 secs. nearly.

(6) A spring balance (an instrument for measuring force, being a spring provided with a scale to show the amount of its elongation) is graduated for a place where $g=32{\cdot}2$, and indicates 1·6 pounds-weight at a place where $g=32$. Find the correct value of the force thus measured.

Ans. 1·61 pounds-weight.

(7) Find the force which must be exerted by a man in an elevator on a body of 1 lb. mass which he holds in his hand, to prevent its moving relatively to the elevator when the elevator is moving (a) with uniform speed, (b) with an upward acceleration of 8 ft.-sec.-units, (c) with a downward acceleration of 8 ft.-sec.-units, (d) with a downward acceleration of 33 ft.-sec.-units.

Ans. (a) 32·2 poundals upwards, (b) 40·2 poundals upwards, (c) 24·2 poundals upwards, (d) 0·8 poundals downwards.

(8) Show that in any motion of a particle the tangential component of the force acting on it may be measured by the rate per sec. at which momentum is increased.

(9) Prove that if W lbs. be acted upon by a uniform force of P pounds-weight for t sec., the velocity acquired will be Pgt/W, and the distance traversed $Pgt^2/(2W)$.

(10) A body of 10 lbs. mass and with an initial velocity of 20 ft. per sec. in a northerly direction is acted upon by two forces, one of 100 poundals in a north-easterly direction and the other of the same magnitude in a north-westerly direction. Find its velocity after 1 min.

Ans. 868·5... ft. per sec. in a northerly direction.

(11) Find the impulse necessary to produce in 20 lbs. a speed of 25 ft. per. sec.

Ans. 500 absolute ft.-lb.-sec. units.

(12) Two particles, each of mass m, are at rest side by side when one is struck a blow of impulse Φ in a given direction, while a constant force F begins at the same instant to act upon the other in the same direction. Prove that if after travelling a distance s in the time t, they are again side by side, $2\Phi = Ft$ and $2\Phi^2 = mFs$.

(13) A particle of mass m is moving in an easterly direction with a velocity v. Find the impulse necessary to make it move in a northerly direction with an equal velocity.

Ans. $mv\sqrt{2}$ in a north-westerly direction.

(14) A particle of mass m moves with uniform speed v in a circle of radius r. Find the force acting upon it.

The particle has an acceleration equal to v^2/r directed towards the centre of the circle (121). Hence the force must be in the same direction and equal to mv^2/r.

[A body moving in a curved path was formerly thought to *have* what was called *centrifugal force*, which required to be neutralized by a force applied to the body (through a string or by other means) towards the centre of curvature (and called therefore centripetal force), in order that the body might be kept in a curved path. Thus a body moving with uniform speed in a circle was considered to be in equilibrium (*i.e.*, to have no acceleration) under equal and opposite forces, the supposed centrifugal force and the actually applied centripetal force. The necessary centripetal force being known to have the magnitude mv^2/r, the centrifugal force was supposed to have that magnitude also. According to our modern conception of force, a body cannot be said to *have* a force. More-

over, we now know that if no force be applied to a body it will move with uniform speed in a straight line, and that, if it is to be made to move in a circle, the resultant force on it must be centripetal. Though the old notion of centrifugal force has been abandoned, the term is still used, being applied by different writers in different ways. It is applied (1) in its original sense, some writers finding it still convenient in some cases to imagine a body moving uniformly in a circle as acted on by a force equal and opposite to the actual centripetal force under which it moves ; (2) to the actual centripetal force under which the body moves ; (3) to the reaction of the moving body on the body by which the centripetal force is exerted, the centrifugal and centripetal forces being thus opposite aspects of the same stress ; (4) to the acceleration of the moving body. Such varying usage leads to great confusion. The old term should be laid aside with the old hypothesis on which it was based.]

(15) Find the horizontal force which must be exerted on an engine of 20 tons which is to go round a curve of 600 yds. radius at the uniform rate of 30 mls. an hour.

Ans. 0·67 ton-weight nearly.

(16) A stone of 4 kgr., attached to a fixed point by a weightless inextensible string 3 metres long, moves uniformly in a circle in the horizontal plane through the fixed point. Find (*a*) the tension in the string when the speed of the stone is 20 cm. per sec., and (*b*) the time of revolution when the tension of the string is equal to the weight of 12 kgr. [We shall investigate farther on (383) the action of forces on bodies through strings. Meantime, we may consider the above string to be a means of keeping the particle at a constant distance from the fixed point and of exerting on it a force, usually called a tension, directed towards the fixed point.]

Ans. (*a*) 5,333$\frac{1}{3}$ dynes ; (*b*) 2 sec. approximately.

(17) A man, standing at one of the poles of a rotating planet, whirls a body of 20 lbs. mass on a smooth horizontal plane by a string 1 yd. long at the rate of 100 turns per minute. He finds that the difference of the forces which he has to exert according as

he whirls the body one way or the opposite is 0·01 pound-weight. Find the period of rotation of the planet.

Ans. 13 h. 37 min. 21·6 sec.

(18) A railway carriage is going round a curve of 500 ft. radius at the rate of 30 mls. per hour. Find how much a plummet hung from the roof by a thread will be deflected from the vertical.

Ans. 6° 51′·4....

(19) A particle of mass m is attached by a massless string of length l to a fixed point, and moves with uniform speed v in a circular path about a vertical axis through the fixed point. Find the tension in the string and the time of a revolution, when the string has a given inclination θ to the axis. [This arrangement is called the *conical pendulum.* The distance h of the fixed point from the plane of the particle's motion is called the height of the pendulum.]

The particle is acted upon by two forces, its weight mg vertically downwards, and the tension in the string T directed towards the fixed point. Its resultant acceleration is $v^2/(l \sin\theta)$ and is directed towards the centre of its path. The sum of the components of the acting forces in this direction is $T \sin\theta$. Hence

$$T \sin\theta = mv^2/(l \sin\theta).$$

The particle has no acceleration in a vertical direction, and the components of the forces in that direction are mg downwards and $T\cos\theta$ upwards. Hence

$$T\cos\theta - mg = 0.$$

From either of these equations T may be found. Eliminating T we obtain

$$v^2 = lg \sin\theta \tan\theta.$$

Hence, if r is the radius of the circular path,

$$v^2 = r^2 g/h.$$

If therefore ω is the angular velocity of the particle about the centre of its path, $\omega = \sqrt{g/h}$, and if t is the time of revolution,

$$t = 2\pi/\omega = 2\pi\sqrt{h/g},$$

which also (187) is the time of oscillation of a simple pendulum of length h.

If θ is indefinitely small, h and l are ultimately equal, and hence $t=2\pi\sqrt{l/g}$ ultimately. Compare this result with that of 190, which shows that in this case the motion is the resultant of two simple harmonic motions whose common period is $2\pi\sqrt{l/g}$.

(20) A particle of mass m, attached by an inextensible string (length $=l$) to a fixed point, moves in a vertical plane through the fixed point in a circle of radius l. Find the tension T of the string in any position.

Let V be the speed at the highest point A of the path, v the speed at any point P, θ the angle subtended at the fixed point by the arc AP. The normal component of the particle's acceleration when at P is v^2/l. Since the vertical distance through which it has fallen from A is then $l(1-\cos\theta)$, we have (185)

$$v^2 = V^2 + 2gl(1-\cos\theta).$$

Hence the normal acceleration

$$v^2/l = V^2/l + 2g(1-\cos\theta).$$

The forces acting on the particle at P are the tension T towards the fixed point and the weight of the particle mg downwards. The sum of the components towards the fixed point is $T+mg\cos\theta$. Hence

$$T+mg\cos\theta = m\{V^2/l + 2g(1-\cos\theta)\},$$

by which equation T is determined.

Show that the least and greatest values of T are $m(V^2/l-g)$ and $m(V^2/l+5g)$ respectively, and that T has these values at the highest and lowest points of the path respectively.

Show also that the least value of V with which a circle will be described is $\sqrt{lg}$, and that, when V has this value, the greatest value of T is equal to six times the weight of the particle.

(21) A particle moving in a straight line is acted upon by a force directed towards a fixed point in the line and proportional to the distance of the particle from it. Show that the particle's motion is simple harmonic, and that, if f is the force on the particle when

at unit distance from the fixed point, the period of its simple harmonic motion is $2\pi\sqrt{m/f}$, m being its mass.

(22) A body of 7 lbs. mass, hung from a fixed support by a massless spiral spring, and vibrating in a vertical line, makes 80 complete vibrations per minute. What force will the spring exert when extended 2 inches? [The force exerted by a compressed or extended spiral spring is proportional to the amount of the compression or extension.]

Ans. 81·9 poundals approximately.

(23) A particle moves in an ellipse under a force directed towards one of the foci. Show that the force is inversely proportional to the square of the distance of the particle from the focus.

(24) A particle of mass m slides down a smooth inclined plane (inclination $=\theta$), its motion being opposed by a force F, inclined to the plane at the angle ϕ. Find (a) the acceleration, and (b) the reaction of the plane. [A smooth body is one which reacts upon another body in contact with it in a direction normal to its surface at the point of contact. Smooth bodies are of course purely ideal. The stresses between actual bodies in contact are not in general normal to the surface. We shall see farther on (328) how their directions are determined.]

Ans. (a) $F\cos\phi/m - g\sin\theta$, up the plane; ($b$) $mg\cos\theta - F\sin\phi$.

(25) A particle slides down a smooth curve in a vertical plane, starting from rest at a given point. If the curve have such a form that at every point the resultant force on the particle is equal to its weight, the radius of curvature at any point will be twice the intercept of the normal to the curve at that point between the curve and the horizontal line through the starting point.

(26) A particle (mass $=m$) slides in a vertical plane down the edge of a smooth circular disc (radius $=r$) whose axis is horizontal. Show that if it start from rest at the highest point, it will quit the disc after describing an arc subtending at its centre an angle $\cos^{-1}\frac{2}{3}$.

Let v be the speed of the particle after describing an arc sub-

tending at the centre an angle θ, then (Ex. 20) the normal acceleration is

$$v^2/r = 2g(1-\cos\theta).$$

The forces acting on the particle are the reaction R of the disc, normally outwards, and its weight mg; and the sum of their normal components is $mg\cos\theta - R$. Hence

$$mg\cos\theta - R = 2mg(1-\cos\theta);$$

and $$R = mg(3\cos\theta - 2).$$

Hence, for $\theta = \cos^{-1}\frac{2}{3}$, $R=0$; and for $\theta > \cos^{-1}\frac{2}{3}$, R is negative, *i.e.*, the disc must attract the particle if they are to remain in contact.

(27) A particle slides down a smooth cycloid placed in a vertical plane, with its vertex upwards and base horizontal. It starts from rest at the vertex. Show that it will leave the curve at the point where the horizontal line drawn through the centre of the generating circle cuts the curve. [If from a point P of a cycloid a normal be drawn meeting the base in the point S, the radius of curvature at P is equal to $2PS$.]

321. *Impact.*—When two bodies in relative motion come into contact, they are said to impinge upon one another or to undergo impact. The consequence of the impact is a change in their velocities. Hence during the impact a stress must have acted between the bodies; and in applying the equation of motion it is often necessary that we should have some means of determining the stress.

In actual bodies the stress is usually of very short duration, and it is thus more convenient to determine the impulse of the stress than the stress itself. In all cases it affects only the component velocities of the impinging bodies in its own direction. In some cases it is of sufficient magnitude only to equalize these component velocities; in others its magnitude is such as to make the bodies recoil, or move away from one another, after impact. Whether or not particles would behave, on im-

pinging, like actual bodies, we have no means of knowing. For the purpose of illustrating the subject by problems, we may assume that they would.

At our present stage we have to consider only the special case of a particle impinging upon a smooth surface of a fixed body. In that case the direction of the stress is normal to the surface. If u and u' are the components normal to the surface, of the particle's velocity just before and just after impact, u is called the velocity of approach and u' that of recoil. Now the stress must be sufficient to change a velocity of approach u into a velocity of recoil u' *i.e.*, if m is the mass of the particle, to produce a change of momentum equal to $mu+mu'$. Hence, if Φ is the impulse,

$$\Phi = mu + mu'.$$

If e is the ratio of the velocity of recoil to that of approach, $e = u'/u$. Hence

$$\Phi = mu(1+e).$$

If ϕ is the stress which is just sufficient to destroy the velocity of approach, and produces no recoil, we have $\phi = mu$. Hence

$$\Phi = \phi(1+e).$$

Newton found by experiment (379) that with given impinging bodies the ratio of the velocity of recoil to that of approach is constant. The ratio e is therefore called the *coefficient of restitution* for the given bodies.

322. *Examples.*

(1) A body of 4 lbs. mass, moving with a velocity of 10 ft. per sec. in a direction inclined 60° to a smooth surface, impinges upon and is reflected by that surface, the coefficient of restitution being 0·5. Find the impulse of the stress.

Ans. $30\sqrt{3}$ absolute ft.-lb.-sec. units.

(2) A particle impinges on a smooth plane, the coefficient of restitution being e, the angle between the direction of the particle's motion before impact and the normal to the plane (called the angle of incidence) being α, and the angle between the direction of its motion after impact and the normal (called the angle of reflection) being θ. Show that

$$\tan\theta/\tan\alpha = 1/e.$$

[Let u, v be the components of the particle's velocity normal and parallel respectively to the given plane before impact; u', v the same quantities after impact. Then $v/u = \tan\alpha$ and $v/u' = \tan\theta$. And $u' = eu$.]

(3) A particle of mass m is let fall from a height h upon a smooth horizontal plane and rebounds to a height h'. Find (a) the impulse of the stress, and (b) the coefficient of restitution.

Ans. (a) $m\sqrt{2g}(\sqrt{h}+\sqrt{h'})$; ($b$) $\sqrt{h'/h}$.

(4) Prove that the velocity of a particle moving on a smooth horizontal plane is reversed in direction after impinging successively on two fixed smooth vertical planes at right angles to one another, the coefficients of restitution being the same for both planes.

(5) A particle is projected from a point A in the circumference of a circle and after impinging at three other points in the circumference returns to A. Show that the tangents of the four angles of incidence are $e^{\frac{3}{2}}$, $e^{\frac{1}{2}}$, $e^{-\frac{1}{2}}$, and $e^{-\frac{3}{2}}$, e being the coefficient of restitution.

(6) A ball falls vertically from rest for 1 sec. and then strikes a smooth plane inclined 45° to the horizon, the coefficient of restitution being 1. Show that it will again strike the plane in 2 sec.

(7) A particle, after sliding from rest for $4/\sqrt{3}$ sec. down a smooth plane inclined 60° to the horizon, strikes a horizontal plane (coefficient of restitution $= \frac{1}{3}$) and rebounds. At what distance will it again strike this plane?

Ans. 37·18... ft.

(8) A ball is projected at an elevation α towards a smooth vertical wall (coefficient of restitution $= e$) from a point whose

distance from the wall is a. What must the velocity of projection be that the ball may return after its rebound to the point of projection?

Ans. $[ga(1+e)/(e \sin 2\alpha)]^{\frac{1}{2}}$.

(9) A particle is projected from a point on an inclined plane of inclination α, with a velocity v, inclined β to the plane. Find the time between the n^{th} and the $(n+1)^{\text{th}}$ rebounds, the coefficient of restitution being e.

Ans. $2e^n v \sin \beta/(g \cos \alpha)$.

(10) A ball is projected with a velocity of given magnitude from a given point in a smooth plane inclined α to the horizon (coefficient of restitution $=e$). Find the direction of the velocity that the ball may cease to hop just as it returns to the point of projection. [The ball ceases to hop after an infinite series of hops. Express the time in which the ball returns to the point of projection (1) by the aid of the last example, noting that

$$1/(1-e)=1+e+e^2+\text{etc.};$$

and (2) by considering the component motion parallel to the plane; and equate these expressions.]

Ans. The inclination of the velocity to the inclined plane is $\cot^{-1}[\tan \alpha/(1-e)]$.

(11) A ball is projected from a point in a plane of inclination α (coefficient of restitution $=e$), with a velocity V at right angles to the plane. Find its distance from the point of projection when it ceases to rebound.

Ans. $2V^2 \sin \alpha/[g(1-e)^2 \cos^2 \alpha]$.

(12) A stream of particles, each of mass m grm., moving in the same direction with a velocity u cm. per sec., impinge successively on a fixed plane (coefficient of restitution $=e$) inclined α to the direction of their velocity. If n particles reach the plane per sec., find the force exerted by the plane. [The plane exerts on the particles a series of impulses. The force exerted is the sum of all the impulses occurring in 1 sec.]

Ans. $mnu(1+e) \sin \alpha$ dynes.

(13) A uniform chain (linear density $=\rho$ lbs. per foot) is held in the hand by one end, its other end being in contact with a horizontal table (coefficient of restitution $=0$). At a given instant it is let go. Show that the force exerted by the table on the chain after t sec. is three times the weight of the portion of the chain then lying coiled on the table. [The various links of the chain having all at any given instant the same velocity fall as though they were unconnected particles. After t sec. $\frac{1}{2}\rho g t^2$ lbs. of chain lie on the table, and the force exerted by the table is also destroying the momentum of $\rho g t$ lbs. of chain per sec.]

323. *Equilibrium.*—A particle is said to be in equilibrium or in a static condition when the forces acting on it produce in it a resultant acceleration equal to zero. The acting forces are also said to be in equilibrium in this case. A particle in equilibrium must therefore either be at rest or be moving with uniform speed in a straight line.

The subject of equilibrium, together with all those portions of Dynamics which are necessary for its discussion are frequently treated separately under the title *Statics*, the other department of Dynamics, which treats of forces as producing acceleration, being then called *Kinetics.* Some writers employ the term Dynamics as synonymous with Kinetics, and apply the term Mechanics to what we have called Dynamics.

324. *Condition of Equilibrium.*—That the resultant acceleration of a particle may be zero, the resultant force acting on it must (317) be zero also. And if the resultant force be zero, the resultant acceleration will be zero also. Hence the vanishing of the resultant force is the necessary and sufficient condition of equilibrium. This condition may be otherwise expressed, viz., that any one of the forces acting on a particle must be equal and opposite to the resultant of all the rest.

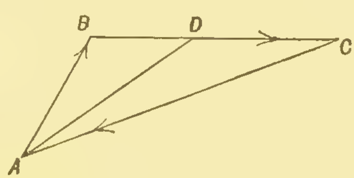
325. *Expressions for Condition of Equilibrium in Special Cases.*—In special cases the following different modes of expressing completely or partially the condition of equilibrium are found convenient in the solution of statical problems.

(*a*) If two forces only act on a particle, they must be equal and opposite.

(*b*) If three forces act on a particle, they must all lie in one plane. For the resultant of any two must be in their plane and must be opposite to the third.

(*c*) If three component forces can be represented by the sides of a triangle taken the same way round, the resultant is zero. This is an immediate inference from the triangle of forces (313).

(*d*) Conversely, if three forces are in equilibrium and if they can be represented in direction by the sides of a triangle taken the same way round, they will also be represented by them in magnitude.—Let AB, BC, CA represent in direction the three component forces P, Q, R respectively, which are in equilibrium. Let AB represent P in magnitude also. If BC does not represent Q in magnitude,

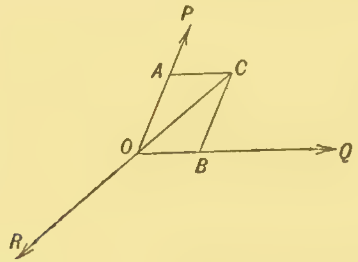


cut off BD from it of the proper length to do so. Then the resultant AD of AB and BD must be in equilibrium with the third force whose direction is CA. That is, two forces whose directions are not in the same straight line may be in equilibrium, which is impossible. Hence the assumption that BC does not represent Q in magnitude was wrong. Similarly it may be shown that CA represents R in magnitude.

(*e*) If three forces P, Q, R are in equilibrium, they are proportional to the sines of the angles between the

directions of Q and R, P and R, and P and Q respectively. OA and OB representing P and Q, R must be represented by CO the diagonal of the parallelogram AB. Now

$$OA : AC : CO = \sin OCA : \sin COA : \sin OAC.$$



If the angle between the directions of P and Q be written $\widehat{PQ}$, we have

$$\sin OCA = \sin(180^\circ - \widehat{QR}) = \sin \widehat{QR},$$
$$\sin COA = \sin(180^\circ - \widehat{PR}) = \sin \widehat{PR},$$
$$\sin OAC = \sin(180^\circ - \widehat{PQ}) = \sin \widehat{PQ},$$

Hence $\quad P : Q : R = \sin \widehat{QR} : \sin \widehat{PR} : \sin \widehat{PQ}.$

(*f*) If more than three forces act, it may be shown from the polygon of forces that, if any number of component forces can be represented by the sides of a polygon taken the same way round, their resultant must be zero. But the converse proposition similar to that of (*d*) does not hold.

326. *Analytical Expression for Condition of Equilibrium.*—We may express the condition of equilibrium of a particle in a way applicable to all cases by employing the analytical expression for the resultant of any number of forces. If F_1, F_2, etc., are the magnitudes of the component forces, α_1, β_1, γ_1, α_2, β_2, γ_2, etc., the angles made by their directions with three fixed rectangular axes, and R their resultant, we have (313 and 90)

$$R = \{(\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2\}^{\frac{1}{2}}.$$

If there is equilibrium, $R=0$. Hence in that case

$$(\Sigma F\cos\alpha)^2+(\Sigma F\cos\beta)^2+(\Sigma F\cos\gamma)^2=0.$$

But if the sum of three essentially positive quantities is zero, the quantities themselves must each be zero. Hence

$$\Sigma F\cos\alpha=\Sigma F\cos\beta=\Sigma F\cos\gamma=0,$$

i.e., the condition that the algebraic sum of the components of the acting forces in each of any three rectangular directions must be equal to zero is a necessary condition of equilibrium. It is evident that it is also sufficient.

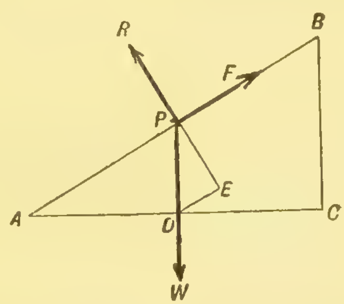
If the forces are coplanar, the angles γ become right angles and the angles β become the complements of the angles α. Hence the equations expressing the condition of equilibrium become

$$\Sigma F\cos\alpha=\Sigma F\sin\alpha=0.$$

327. *Examples.*

(1) A particle of weight W rests upon a smooth inclined plane of inclination α (to the horizontal plane) under a force F acting up the plane (*i.e.*, in the direction of a line of greatest slope). Find the magnitude of F and of the reaction R of the plane.

Let AB be the line of greatest slope of the inclined plane, AC a horizontal line in the vertical plane through AB. Then the particle at P is in equilibrium under the three forces, W acting vertically downwards, R acting at right angles to AB in a vertical plane, and F in the direction AB.



If a perpendicular be let fall from D (the point in which a vertical line through P meets AC) on the direction of R and meet it in E, we form a triangle PDE whose sides PD, DE, and EP taken the same way round have the same directions as the forces W, F, R respectively. Hence (325, *d*) $\qquad W:F:R=PD:DE:EP.$

Now APE being a right angle, the angle DPE is equal to α. Hence $DE = PD \sin \alpha$ and $EP = PD \cos \alpha$. Therefore

$$W : F : R = 1 : \sin \alpha : \cos \alpha.$$

And hence $$F = W \sin \alpha, \text{ and } R = W \cos \alpha.$$

Otherwise thus :—The angle $\widehat{RF}$ is $\pi/2$ radians, the angle $\widehat{RW}$ $(\pi - \alpha)$ radians, and the angle $\widehat{FW}$ $(\pi/2 + \alpha)$ radians. Hence (325, e)

$$W : F : R = \sin (\pi/2) : \sin (\pi - \alpha) : \sin (\pi/2 + \alpha)$$
$$= 1 : \sin \alpha : \cos \alpha.$$

Otherwise thus :—Choose any two directions at right angles to one another and put the algebraic sum of the components of the forces in each of these directions equal to zero (326). To simplify the equations it is well to choose the directions so that they may coincide with those of as many of the forces as possible. In the direction AB we have

$$F - W \sin \alpha = 0.$$

In the direction perpendicular to AB we have

$$R - W \cos \alpha = 0.$$

When the inclined plane is used as a simple machine the ratio of W the weight of the body kept in equilibrium on it to F the force which must be applied to the body for this purpose is called its *mechanical advantage.* Hence in the case in which the force F acts up the plane the mechanical advantage is cosec α.

If from B a perpendicular BC be drawn to AC, BC is called the height of the inclined plane, whose length is AB and base AC. The letters h, l, b are frequently used to denote these lines. Hence in the present case $W/F = l/h$.

(2) (a) Find the mechanical advantage of a smooth inclined plane (length $= l$, height $= h$, base $= b$) when the applied force acts in a horizontal direction, and (b) express the reaction (R) of the plane on a particle in equilibrium on it in terms of the weight (W) of the particle.

Ans. (a) b/h, (b) $R = Wl/b$.

(3) A particle is in equilibrium on a smooth inclined plane (inclination $= \alpha$) under the action of a force F whose inclination to the

inclined plane is θ and to the horizon $(\alpha+\theta)$. Find (a) F, and (b) the reaction of the plane, in terms of the weight W of the particle.

Ans. (a) $W \sin \alpha/\cos\theta$, (b) $W \cos(\alpha+\theta)/\cos\theta$.

(4) A body is kept at rest on a smooth inclined plane by a force acting up the plane, and equal to half the weight of the body. Find the inclination of the plane.

Ans. 30°.

(5) A body is in equilibrium on a smooth inclined plane, and the applied force and the reaction of the plane are each equal to the weight of the body. Determine (a) the inclination of the plane, and (b) the direction of the applied force.

Ans. (a) 60°, (b) inclined 30° to both inclined and horizontal planes.

(6) A body is supported on a smooth inclined plane by a force equal to its weight. Show that the reaction of the plane is double what it would be if the body were supported by the least possible force.

(7) P is the value of the force which, acting up a smooth inclined plane, keeps a body on it in equilibrium. Q is the magnitude of the force necessary to support the body when its direction is such that it is equal to the reaction of the plane. Show that P acting up the plane could just support a body of weight Q on a plane of twice the inclination.

(8) A heavy body of 12 lbs. mass is kept in equilibrium by two applied forces, one horizontal and the other inclined 30° to the horizon. Find the forces.

Ans. Inclined force = 772·8 pdls., the other = 669·2... pdls.

(9) Forces of $2\frac{1}{2}$, 6, and $6\frac{1}{2}$ poundals keep a particle in equilibrium. Show that two of them are at right angles, and find the angle between the greatest and least.

Ans. $\cos^{-1}(-\frac{5}{13})$.

(10) A heavy bead (weight $= W$) capable of sliding on a smooth circular wire in a vertical plane is held at a distance equal to the radius of the circle from its highest point by a force directed to

that point. Find (a) the force, and (b) the reaction of the wire on the bead.

Ans. (a) W, (b) W.

(11) R is the smallest and R' the greatest force which, along with P and Q, can keep a particle in equilibrium. Show that, if P, Q, and a force $(R+R')/2$ keep a particle in equilibrium, two of these forces are equal; and that, if P, Q, and a force $\sqrt{RR'}$ do so, two of them must be at right angles.

(12) Two equal particles, each attracting with a force varying directly as the distance, are situated at the opposite extremities of a diameter of a horizontal circular wire on which a small smooth ring is capable of sliding. Prove that the ring will be kept at rest in any position under the attraction of the particles.

(13) Show that there is but one point in a triangle at which a particle would be in equilibrium if acted upon by forces represented by the lines drawn from it to the angular points of the triangle.

(14) Show that a particle is in equilibrium if acted upon by three forces represented in direction by the perpendiculars from the angular points of a triangle on the opposite sides, and in magnitude by the reciprocals of the lengths of those perpendiculars.

(15) On a smooth inclined plane of inclination $\cos^{-1}\frac{1}{3}$ a particle is in equilibrium under the action of a certain force up the plane. Find the direction in which an equal force must be applied, that it, along with a horizontal force of the same magnitude, may also keep the particle in equilibrium.

Ans. Inclination to inclined plane$=\cos^{-1}\frac{2}{3}$.

(16) Show that a particle is in equilibrium under the following forces:—4, $N.$; 2, $N.30°\,E.$; 4, $E.$; $2\sqrt{3}$, $E.\,30°\,S.$; $4\sqrt{2}$, $S.W.$; $2\sqrt{3}$, $W.\,30°\,S.$; and 2, $N.30°\,W.$

(17) From two points lines are drawn to the angular points of a triangle. Find the condition that a particle acted upon by forces represented by these six lines may be in equilibrium.

Ans. The given points must be on a straight line through the point of intersection of the straight lines drawn from the angular

points of the triangle to the middle points of the opposite sides, and must be at equal distances from this point.

(18) A string whose ends are fixed at two points A and B in the same horizontal line has, knotted at C, another string carrying a heavy body. Compare the tensions in CA and CB, when they are of such length that ACB is a right angle, the whole system being in equilibrium. [We shall prove farther on (389) that when strings are knotted together, the stresses or tensions in them are in general different. In such cases, if there is equilibrium, the knot must be considered to be in equilibrium under the action of the stresses in the strings.]

Ans. As $CB : CA$.

(19) A string has its ends fixed at A and B. Another string is knotted to it at C and supports a body of weight W. The inclinations of CA and CB to the horizon are θ and ϕ respectively. Find the tensions in CA and CB when there is equilibrium.

Ans. $W\cos\phi/\sin(\theta+\phi)$ and $W\cos\theta/\sin(\theta+\phi)$ respectively.

(20) Three strings have one end each knotted together at C. Two of them are attached to fixed points at A and B, and the tensions in them are T and T' respectively. The third supports a particle whose weight is W. Find the inclinations θ and ϕ of CA and CB to the horizon when there is equilibrium.

Ans. $\theta=\sin^{-1}\dfrac{T^2-T'^2+W^2}{2WT}$; $\phi=\sin^{-1}\dfrac{T'^2-T^2+W^2}{2WT'}$.

(21) A string whose length is 10 feet has its ends fastened at two points in a horizontal line 6 feet apart. A small smooth massless ring slides on the string carrying a body weighing 10 lbs. Find the tension in the string when there is equilibrium. [We shall prove farther on (391) that when the direction of a flexible string is changed by its being bent round a smooth body the stress throughout the string is the same. In this problem the portion of the string in contact with the ring is in equilibrium under the action of a force equal to and codirectional with the weight of the body which the ring carries, and of the equal tensions in the two portions of the string.]

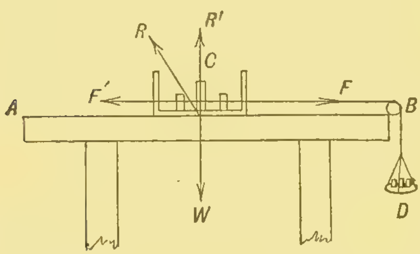
Ans. $201\frac{1}{4}$ poundals.

(22) A fixed smooth hemispherical bowl whose rim is horizontal has, resting inside it, a particle of weight W attached by a string which passes over the rim to another particle of weight W' which hangs freely. Find the position of the particle in the bowl.

Ans. If θ is the angle subtended at the centre of the bowl by the portion of the string within it

$$\theta = 2\cos^{-1}\frac{W' + (W'^2 + 8W^2)^{\frac{1}{2}}}{4W}.$$

328. *Friction.*—We are now able to understand the experimental determination of the direction of the stress between two actual rough bodies in contact with one another. The figure shows the apparatus employed. AB is a horizontal table, C a flat-bottomed box upon it. To C a string is attached which passes over a pulley at B and supports a pan D. Weights (*i.e.*, standards of mass) are placed in C and D. Before the string is attached C remains motionless on the table. The only forces acting on it are its weight and the reaction of the table. Hence this reaction must be vertical and therefore normal to the surface of contact between C and the table. If now the string be attached and weights added gradually to D, C remains motionless until they reach a certain amount. For all loads in D less than this amount, C is in equilibrium* under three forces—its weight W, the reaction of the table R, and the force F exerted by the string (equal to the weight of D and in the direction of the string). Hence R must be in the plane of F and W, and so inclined to W, which is normal to the surface of contact, that it has a component F' equal and opposite to F.



*The motion of the box is (454) the same as that of a particle acted upon by the same forces, provided the box undergo translation only.

This component F' resists the sliding of the box over the surface of the table, and is called the *friction* between the box and the table. It increases with F until the box is just on the point of moving, when it has its greatest value and is called the *limiting static friction.* If we increase F still more the box begins to move with an acceleration; and the greater we make F, the greater is the acceleration. If the acceleration be observed, the resultant horizontal force may be determined, and the difference between this force and F is the value of F' in this case. The value of F' when the box is in motion is called the *kinetic friction.* It is found (by more refined experimental methods than the above) to be usually slightly less than the limiting static friction and to be (at any rate very nearly) independent of the velocity of the box if the velocity is not great.

If weights of different amounts are put into the box C it is found that the friction (whether limiting static or kinetic) is, within limits, proportional to the weight of the box and its contents, and therefore to the normal component of the reaction. If boxes of the same substance and weight, but with bottoms of different areas, are used, the friction is found to be independent of the area of the surface of contact. If the substance of the bottom of the box and that of the table, or their state of surface, be changed, the friction is found to change also.

If F' is the value of the friction (whether limiting static or kinetic), and R' the normal component of the reaction R, we have thus $F' = \mu R'$, where μ is a constant for two bodies of given substances with their surfaces of contact in given states. It may be determined by such experiments as the above, and it has different values according as relative motion of the one body over the surface of the other is on the point of occurring or is actually occurring, being usually slightly greater in the former case than in the latter. In the former case, μ is

called the coefficient of static friction; in the latter, that of kinetic friction.

The inclination to the normal of the reaction R of the bodies in contact may be expressed in terms of the coefficient of friction. R' and F' being the normal and frictional components of R, we have, if ϵ is the inclination of R to the normal, tan $\epsilon = F'/R' = \mu$, and $\epsilon = \tan^{-1}\mu$.

If μ is the coefficient of static friction the value of ϵ thus determined is the greatest possible inclination of the reaction to the normal. It is called the *angle of repose.* As R is in the plane containing the normal to the surface of contact, and the direction in which the acting forces tend to produce sliding, and as this direction may be any whatever in the tangent plane at the point of contact of the bodies, the direction of the reaction when sliding is on the point of occurring may be any line on the surface of a cone whose axis is the normal at the point of contact, and whose semi-vertical angle is the angle of repose. The direction of the reaction under all circumstances must be included in this cone.

The ideal *perfectly rough* body is one over whose surface sliding is impossible. In the case of such a body the reaction is supposed to have any direction and magnitude that may be necessary to prevent sliding.

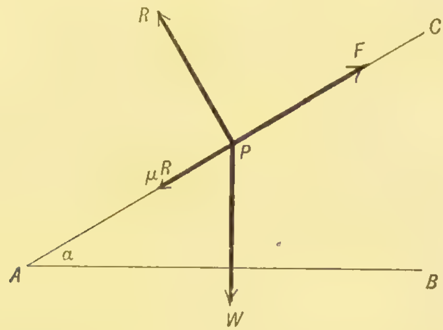
The above statement of the laws of friction is sufficient for our purpose. For a more detailed statement of our knowledge of this subject, see recent works on engineering.

329. *Examples.*

(1) A particle of mass m is moving up an inclined plane (coefficient of friction $= \mu$) of inclination α, being acted upon by a force F up the plane. Find its acceleration and the reaction of the plane.—Let R be the normal component of the reaction of the plane. Then μR is the component in the plane, and as the point is moving

up the plane, μR is directed down it. Hence, if a is the acceleration up the plane,

$$a=(F-\mu R-W\sin \alpha)/m.$$



As there is no acceleration normal to the plane,

$$0=R-W\cos \alpha.$$

Hence $$a=[F-W(\mu\cos \alpha+\sin \alpha)]/m.$$

Also the resultant reaction is (313 and 86, V)

$$R\sqrt{1+\mu^2}=W\sqrt{1+\mu^2}\cos \alpha.$$

In the above formulae μ is the coefficient of kinetic friction.

(2) A body of 100 lbs. mass, moving on a horizontal surface with a speed of 10 ft. per sec., comes to rest in 2 sec. Find the coefficient of kinetic friction (supposed independent of the velocity).

Ans. 0·15....

(3) A mass of 100 lbs. is moved along a horizontal plane by a constant horizontal force of 20 lbs.-weight. Determine the displacement in 10 sec., the coefficient of kinetic friction being 0·17.

Ans. 48·3 ft.

(4) A force equal to the weight of 28 lbs. is required to draw a mass of 30 lbs. up a plane inclined 30° to the horizon. Find (a) the coefficient of friction ; (b) the force that would be necessary if the inclination were 45°.

Ans. (a) 0·5... ; (b) $45/\sqrt{2}$ lbs.-weight approximately.

(5) A train is going up an incline of 1 in 70, at the rate of 10 mls. per hour, the friction being equivalent to a force of 8 lbs.-

weight per ton of the train's mass. The incline is 500 ft. in length, and when the train is half way up, a coupling-chain breaks. Find (*a*) how far the train will go up the incline, and (*b*) its speed at the foot of the incline.

Ans. (*a*) 187·05... ft.; (*b*) 17·36... ft. per sec.

(6) A particle impinges on a fixed rough plane (coefficient of friction $=\mu$, that of restitution $=e$) with a velocity v inclined a to the normal. Find (*a*) the magnitude, and (*b*) the inclination to the normal, of the velocity after impact. [The frictional impulse is equal to μ times the normal impulse.]

Ans. (*a*) $\{e^2v^2\cos^2\alpha+[v\sin\alpha-\mu v(1+e)\cos\alpha]^2\}^{\frac{1}{2}}$;

$$(b)\ \tan^{-1}\left[\frac{1}{e}\tan\alpha-\frac{\mu}{e}(1+e)\right].$$

(7) A particle is in equilibrium on a rough inclined plane of inclination α, being just prevented from moving down by a force F acting up the plane. The coefficient of static friction being μ, find F and the reaction of the plane.

This problem may be solved by means of the result of Ex. 1. That the particle may be in equilibrium on the plane we must have $a=0$. Hence

$$F-W(\mu\cos\alpha+\sin\alpha)=0.$$

In the formula of Ex. 1, μ was the coefficient of kinetic friction. When we make $a=0$, it becomes the coefficient of static friction. Also that formula was obtained on the assumption that μR acts down the plane, and therefore that the particle moves, or tends to move, up the plane. If we make μ negative and thus obtain

$$F-W(\sin\alpha-\mu\cos\alpha)=0,$$

we reverse the direction of μR, *i.e.*, we get a formula applicable to the case in which μR acts up the plane, and the particle therefore is prevented by friction from moving down the plane.

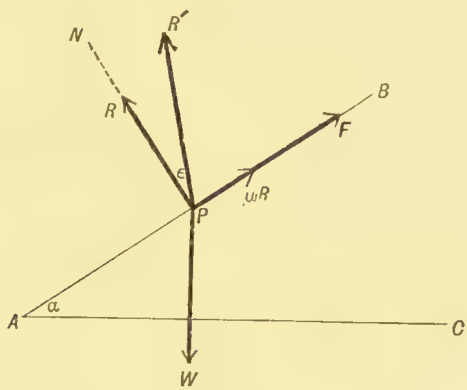
Otherwise thus :—The particle is acted upon by three forces, its weight W, F, and the reaction of the plane R'. R' is inclined to the normal PN at the angle of repose (ϵ), the angle being measured towards PB, because the body is on the point of moving down the plane. Since

$$\widehat{WR'}=180^\circ+\epsilon-\alpha,\ \widehat{R'F}=90^\circ-\epsilon, \text{ and } \widehat{WF}=90^\circ+\alpha,$$

we have $$F:W:R'=\sin(\alpha-\epsilon):\cos\epsilon:\cos\alpha;$$

and hence $$F=W\sin(\alpha-\epsilon)/\cos\epsilon,$$

and $$R'=W\cos\alpha/\cos\epsilon.$$



Otherwise:—Replacing R' by its normal component R and its frictional component μR (up the plane), and resolving in and perpendicular to the direction of AB, we have

$$F+\mu R-W\sin\alpha=0,$$

and $$R-W\cos\alpha=0.$$

Hence $$F=W(\sin\alpha-\mu\cos\alpha),$$

and $$R'=R\sqrt{1+\mu^2}=W\cos\alpha\sqrt{1+\mu^2}.$$

Recollecting that $\epsilon=\tan^{-1}\mu$, it is easy to show the consistency of the above results. The same equations may be obtained in other ways. [See 327 (1).]

(8) A body is in equilibrium on a rough inclined plane of inclination α, under a force F, inclined at the angle θ to the inclined plane. Find the ratio of the weight of the body to the force F (a) when the body is on the point of moving up the plane; (b) when it is on the point of moving down.

Ans. (a) $\dfrac{\cos\theta+\mu\sin\theta}{\sin\alpha+\mu\cos\alpha}$; ($b$) $\dfrac{\cos\theta-\mu\sin\theta}{\sin\alpha-\mu\cos\alpha}$.

(9) Prove that the horizontal force which will just sustain a heavy particle on a rough inclined plane will sustain the particle on a smooth inclined plane provided its inclination is less than that of the rough plane by the angle of repose.

(10) Show that the least coefficient of friction that will allow of a heavy body's being just kept from sliding down an inclined plane of inclination α, the body (weight $= W$) being sustained by a given horizontal force F, is $(W \tan \alpha - F)/(F \tan \alpha + W)$.

(11) A heavy body is kept at rest on a given inclined plane by a force making a given angle with the plane. Show that the reaction of the plane when it is smooth is a harmonic mean between the normal components of the greatest and least reactions when it is rough.

(12) A bead, capable of sliding on a rough circular wire (radius $= r$, coefficient of friction $= \mu$) in a vertical plane, is in equilibrium in the highest position (not being the highest point of the wire) in which equilibrium is possible. Find its position.

Ans. Its distance from the highest point of the circle measured along the circumference is $r \tan^{-1} \mu$.

(13) Show that it is easier to lift a body a given height than to drag it up an inclined plane of that height by a rope parallel to the plane, if the coefficient of friction is greater than the ratio of the difference between the length and height of the plane to its base.

330. *Work done.*—Work is said to be done by a force on a body when its place of application has a component displacement in the direction of the force, and by a body against a force when the place of application of the force has a component displacement in the direction opposite to that of the force. Work may in both cases be said to be done by the force if in the one case it is considered as positive and in the other as negative.

If the force doing the work is uniform, the work done is measured by the product of the force into the component in its direction of the displacement of its point of

application. If W, F, s, a represent the work done, the force acting, the displacement, and the inclination of the directions of the displacement and the force, we have thus, by definition,

$$W \propto Fs \cos a.$$

The work done is therefore measured also by the product of the displacement into the component in its direction of the acting force.

If the force doing the work is variable, the motion of the body may be supposed to be broken up into a large number of small displacements during each of which the force may be considered constant, and the work done is the sum of all the quantities of work done during these small displacements.

331. *Measurement of Work done.*—If we write l for the component of the displacement in the direction of the force, we have $W = kFl$, where k is a constant whose value will depend upon the units involved in the other quantities. We have already selected units of force and length. We can give k the convenient value unity therefore only by properly selecting the unit of work. If $W = 1$, $F = 1$, $l = 1$, k will be equal to 1. Hence we take as unit of work the work done when under the action of unit force a particle has a component displacement of 1 unit in the direction of the force. This derived unit of work will of course vary with the units chosen as simple units.

F. P. S. Gravitational System—The weight of the pound being the unit of force, the unit of work is the work done when a body under this force moves through a distance of 1 foot in its direction. This unit is very largely used in Engineering. It is called the *foot-pound*, and is usually defined as the work done in lifting one pound one foot vertically.

M. K. S. Gravitational System.—The weight of the kilogramme being the unit of force, the unit of work is that done when under this force a body moves through 1 metre in its direction. It is largely used by French engineers, and is called the *kilogramme-metre.* The kilogramme-metre is equivalent to 7·2331 foot-pounds.

F. P. S. Absolute System.—The unit of work is the work done when under a force of 1 poundal, a body moves through 1 foot in the direction of the force. This unit is called the *foot-poundal.* It is clear that, as the weight of 1 lb. is g times the poundal, the foot-pound must be g times the foot-poundal.

C. G. S. Absolute System.—The unit of work is the work done when under a force of 1 dyne a body moves through 1 centimetre in the direction of the force. It is called the *erg.* The *joule* is 10,000,000 ergs, and is equivalent to nearly $\frac{3}{4}$ of a foot-pound. It is the unit of work of the absolute metre-kilogramme-second system.

332. *Dimensions of Unit of Work.*—From the equation $W \propto Fl$, we deduce, as in 300 and 303, $[W] \propto [F][L]$ and $[W] \propto [M][L]^2[T]^{-2}$. A knowledge of the dimensions of units of work is applied in the solution of problems in exactly the same way as in the case of units of speed and rates of change of speed.

333. *Rate of Work, Activity, or Power.*—The *mean rate* at which a force does work in a given time is the quotient of the work done in the time by the time. In general the mean rate varies with the interval of time to which it applies. In any case in which it does not, the rate of doing work is said to be uniform.

The *instantaneous rate* at a given instant is the mean rate between that instant and another when the interval of time between them is made indefinitely small. It has (295) in all cases a finite value.

The rate at which an agent (*e.g.*, a steam engine) can do work is called its *Power* or *Activity*.

334. Let W be the work done by a force F on a particle of mass m in a short time t, and R the rate at which the work is done. Then $R=W/t$. If s is the distance traversed by the body in the direction of F during t, $R=Fs/t=Fv$, where v is the component of the instantaneous velocity of the particle in the direction of F. If a is the instantaneous acceleration produced in the particle by F, we have $F=ma$, and therefore $R=mav$, whence $a=R/(mv)$, *i.e.*, the acceleration produced in a particle by a force working at the rate R, is equal to the quotient of this rate by the momentum of the particle in the direction of F.

If the work is done against a force F', which has a direction opposite to that of F, and produces an acceleration a', the resultant acceleration is $a-a'=R/(mv)-F'/m$. As v increases a decreases. When $a=a'$ there is no resultant acceleration and v becomes uniform and has its greatest value. Hence the greatest velocity which a force working at the rate R can produce against an opposing force F' is equal to R/F'.

335. *Measurement of Power or Rate of Work.*—We have by definition $R=W/t$. When $W=1$ and $t=1$, $R=1$. Hence unit rate of work is unit of work per unit of time. The following are therefore the units of rate of work in the various systems.

F. P. S. Gravitational System—One foot-pound per second.—The unit employed by English engineers is a multiple of this, viz., 550 ft.-pounds per sec., or 33,000 ft.-pounds per min., which is called the *horse-power*.

M. K. S. Gravitational System—One kilogramme-metre per second.—The unit practically employed by

French engineers is 75 kilogramme-metres per sec. (equivalent to 542·486 ft.-pounds per sec.), which is called the *force de cheval.*

F. P. S. Absolute System—One ft.-poundal per sec.

C. G. S. Absolute System—One erg per sec.—A multiple of this unit, viz., 10,000,000 ergs per sec. (equivalent to nearly $\frac{3}{4}$ ft.-pound per sec.) is extensively employed in electrical work. It is called the *watt*, and is equal to the unit of rate of work of the metre-kilogramme-second absolute system.

The dimensions of units of rate of work can be readily shown from the formulae of 334 and 332 to be $[F][L][T]^{-1}$ or $[M][L]^2[T]^{-3}$.

336. *Examples.*

(1) Reduce 50 ergs to kilogramme-metres.

Ans. $5·09 \times 10^{-7}$ approx.

(2) Reduce 20 foot-pounds to ergs.

Ans. $2·712 \times 10^8$ approx.

(3) Show that 1 foot-poundal = 421,390 ergs.

(4) Find the multiplier by which ergs are reduced to foot-pounds.

Ans. $7·37 \times 10^{-8}$.

(5) The second and the foot being the units of time and of length respectively, determine the unit of mass that the derived unit of work may be equal to the foot-pound.

Ans. 32·2 lbs.

(6) The units of mass, work, and length being taken as fundamental units, find the dimensions of the derived unit of time.

Ans. $[L][M]^{\frac{1}{2}}[W]^{-\frac{1}{2}}$.

(7) A man weighing 168 lbs. climbs a mountain 11,000 feet high in 7 hours, the difficulties of the way being equivalent to the carrying of an additional weight of 42 lbs. Show that he has worked at $\frac{1}{6}$ horse-power.

(8) A boy drags a body of 50 lbs. mass on a smooth horizontal plane, doing work upon it at the rate of $\frac{1}{10}$ horse-power. Find its acceleration when its speed is 1 mile per hour.

Ans. 24·15 ft.-sec. units in the direction of motion.

(9) An engine is employed in lifting vertically a bale of goods weighing 1 cwt. (*a*) If the engine is working at 5 H.-P. and the bale has a speed of 5 ft. per sec., find its acceleration. (*b*) At what H.-P. must the engine work to lift the bale with a uniform speed of 1 ft. per sec.?

Ans. (*a*) 125·9 ft.-sec. units upwards; (*b*) $\frac{1}{5}$ H.-P. approx.

(10) A train weighing 75 tons ascends an incline of 1 in 800 with a uniform speed of 40 miles per hour. Assuming the friction to be equivalent to a force of 6 pounds-weight per ton of the train's mass, find the rate at which the engine is working.

Ans. 70·4 H.-P. approx.

(11) Find the greatest speed an engine of 100 H.-P. can give a train of 70 tons mass on an incline of 1 in 100, friction being equivalent to a force of 8 pounds-weight per ton.

Ans. 17·62... miles per hour.

(12) Reduce 20 horse-power to ergs per second.

Ans. $1·492 \times 10^{11}$.

(13) In 1 *force de cheval* how many ergs per second?

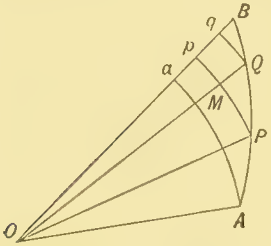
Ans. $7·36 \times 10^{9}$.

(14) If the acceleration of a falling body be taken as unit of acceleration, 1 ton as unit of mass, 1 horse-power as unit rate of work, and 1 minute as unit of time, find the derived unit of length.

Ans. 14·7... feet.

337. *Determination of Work done under given Forces.* (1) *Under a Uniform Force.*—If a particle undergo any motion under a uniform force, no difficulty arises in determining the work done. It is simply the product of the magnitude of the force into the component displacement in its direction.

338. (2) *Under a Central Force, i.e.*, a force directed towards a centre and varying with the distance from the centre.—Let O be the centre of force, AB any path of a particle from A to B, and PQ any indefinitely small portion of the path. Join OA, OP, OQ, OB, and from O as centre describe arcs of circles Aa, Pp, Qq, M being the point of intersection of Pp with OQ.



PQ being small, the force on the particle between P and Q may be considered constant. Let F be its magnitude. QO may be considered its direction. Hence the work which must be done in moving the particle from P to Q is $F \,.\, PQ \cos MQP$, which, since PM and PQ may be considered straight lines and PM is at right angles to OQ, is equal to $F \,.\, MQ$. Now p and q being at the same distance from O as P and Q respectively, the force on the particle, if taken from p to q, would be F also, and $MQ = pq$. Hence the work which must be done in moving the particle from P to Q is the same as that necessary to move it from p to q. We may treat every element of the path in the same way. Hence, by summation, the work necessary to move the particle from A to B is equal to that necessary to move it from a to B in a straight line.

Hence also the work done in moving a particle from A to B is independent of the path, and depends only on the initial and final distances of the points.

339. (a) *The Force directly proportional to the Distance of the Particle from the Centre.*—Let f be its value at unit distance. Then its values at a and B are $f \,.\, Oa$ and $f \,.\, OB$ respectively. Hence its mean value per unit distance between a and B is $\frac{1}{2}f(Oa + OB)$; and consequently the work which must be done in moving the particle from a to B, and therefore from A to B, is

$$\tfrac{1}{2}f(Oa + OB)(OB - Oa) = \tfrac{1}{2}f(OB^2 - Oa^2) = \tfrac{1}{2}f(OB^2 - OA^2).$$

If r and R are the initial and final distances respectively, the work done is $\frac{1}{2}f(R^2-r^2)$.

340. (*b*) *The Force inversely proportional to the Square of the Distance of the Particle from the Centre.*—Let f be its value at unit distance. Then its values at p and q are f/Op^2 and f/Oq^2 respectively. Since pq is indefinitely small, the value of the force between p and q may be put equal to either or to the intermediate value $f/(Op \,.\, Oq)$. Hence the work which must be done in moving the particle from p to q is *

$$\frac{f}{Op \,.\, Oq}(Oq-Op)=f\left(\frac{1}{Op}-\frac{1}{Oq}\right).$$

Let the line aB be divided into the indefinitely small portions (or elements) ap_1, p_1p_2, etc., $p_{n-1}B$. Then, adding together the amounts of work done throughout all the elements of aB, the work done in moving the particle from a to B, and therefore from A to B, is

$$f\left\{\left(\frac{1}{Oa}-\frac{1}{Op_1}\right)+\left(\frac{1}{Op_1}-\frac{1}{Op_2}\right)+\text{etc.}+\left(\frac{1}{Op_{n-1}}-\frac{1}{OB}\right)\right\}$$

$$=f\left(\frac{1}{Oa}-\frac{1}{OB}\right)=f\left(\frac{1}{OA}-\frac{1}{OB}\right)=f\left(\frac{1}{r}-\frac{1}{R}\right),$$

if r and R are the initial and final distances respectively.

Hence also the work done in moving the particle from a point at distance r to a point at an infinite distance from the centre is f/r.

* The error involved in using this expression must be less than

$$f\left(\frac{1}{Op^2}-\frac{1}{Oq^2}\right)(Oq-Op),$$

and therefore less than

$$f\cdot\frac{Oq+Op}{Op^2\,.\,Oq^2}(Oq-Op)^2,$$

and therefore negligible.

341. *Examples.*

(1) Find the work done by the weight of a body of 20 lbs. mass during the first three seconds of its fall from rest.

Ans. 93,315·6... ft.-poundals.

(2) A body of 80 lbs. mass is projected along a rough horizontal plane (coefficient of friction $=0\cdot25$) with a speed of 50 ft. per sec. Find the work done against friction in 1 sec.

Ans. 919·5 ft.-pounds.

(3) Show that the work done in drawing a heavy body up a rough inclined plane is the same as if the body were drawn along the equally rough base and then lifted through the vertical height.

(4) The distance from X to Y is 105 miles, and there are 27 intermediate stations. Train A stops at all stations. Train B runs through without stopping. The average resistances to A and B with the brakes off are equal to $\frac{1}{280}$ and $\frac{1}{224}$ of their respective weights. With the brakes on, the resistances are in both cases equal to $\frac{1}{28}$ of the respective weights. Suppose the brakes to be always applied when the speed has been reduced to 30 miles per hour and not before, find which train is the more expensive and by how much per cent.

Ans. Train A, by 9·4... per cent.

(5) A particle of mass m moves in a circular path of radius r, (a) with uniform speed, (b) with uniform rate of change of speed f. Find the work done in both cases during the motion of the particle through a semicircle.

Ans. (a) none, (b) πmrf.

(6) Show that in the case of a particle which is oscillating with a simple harmonic motion, the work done during its motion from its extreme position to its mean position is twice that done during its motion from a distance equal to $\frac{3}{4}$ of its amplitude to a distance equal to $\frac{1}{4}$ of its amplitude.

(7) A particle weighing $\frac{1}{10}$ oz. has a simple harmonic motion of 0·5 sec. period. Find the work done during the motion from a distance of 3 inches to a distance of 1 inch.

Ans. 0·0274 foot-poundals.

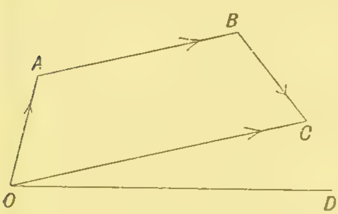
(8) Find the work done by the sun's attraction during the motion of the earth from Aphelion to Perihelion. Mass of earth $= 6{\cdot}14 \times 10^{27}$ grammes, mass of sun $= 324{,}000$ times that of earth, distance at Aphelion $= 1{\cdot}512 \times 10^{13}$ cm., distance at Perihelion $= 1{\cdot}462 \times 10^{13}$ cm. (See 316, Ex. 12.)

Ans. $1{\cdot}79 \times 10^{39}$ ergs.

(9) At the three corners A, B, C, of a square $ABCD$ (side $= 100$ metres) are material particles of 3,928, 7,856, and 11,784 grammes respectively. Find the work done against the gravitational attraction of the particles in moving 1 gramme from the centre to the fourth corner.

Ans. $7{\cdot}82 \times 10^{-8}$ erg approx.

342. *Relation of Work done by Component Forces to that done by Resultant.*—The work done by a force during any displacement of a particle is equal to the sum of the quantities of work done by its components.—Let



OC be the force, OA, AB, BC, its components, whose directions may be any whatever. Let OD be the displacement. By 8 (footnote), θ, α, β, γ being the inclinations of OC, OA, AB, and BC respectively to OD,

$$OC \cos\theta = OA \cos\alpha + AB \cos\beta + BC \cos\gamma.$$

Multiplying by OD we obtain

$$OC.OD \cos\theta = OA.OD \cos\alpha + AB.OD \cos\beta + BC.OD \cos\gamma,$$

by which the proposition is proved.

If F_1, F_2, etc., R, denote the component and resultant forces respectively, d_1, d_2, etc., r, the component displacements in the directions of the forces respectively, the above may be written

$$F_1 d_1 + F_2 d_2 + \text{etc.} = \Sigma Fd = Rr,$$

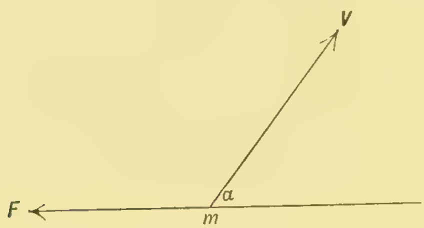
care being taken that, where F and d have opposite directions, the product must have the negative sign.

343. *Energy.*—We have seen that work is said to be done by a body against a force which is acting on it, when it undergoes a displacement having a component in a direction opposite to that of the force. When a particle is thus able to do work it is said to possess work-power or energy.

Energy being power of doing work is measured in terms of the unit of work.

344. *Kinetic Energy.*—A particle which has a velocity is able to do work against a force which has a component in a direction opposite to its velocity. It is therefore said to possess kinetic energy. Kinetic energy is thus work-power due to the possession of a velocity.

To find the kinetic energy of a particle we determine the work done by it against any force during a given diminution of its velocity.—Let the particle of mass m have an initial velocity V, and let it do work against a constant force F. It will undergo a displacement having a component in a direction opposite to that of F. Let that component be d and let the velocity of the particle be reduced to v. Let the inclination of V to F's line of action be α. Then the particle has in that line a component initial velocity $V\cos\alpha$ and an acceleration $-F/m$, and at right angles to it an initial velocity $V\sin\alpha$ and no acceleration. Hence, after the displacement, its component velocity u in F's line of action is such that



$$u^2 - V^2\cos^2\alpha = -2Fd/m.$$

Its resultant velocity v is such that

$$v^2 = u^2 + V^2\sin^2\alpha.$$

Hence $$v^2 - V^2 = -2Fd/m,$$
and $$Fd = \tfrac{1}{2}mV^2 - \tfrac{1}{2}mv^2.$$

If the force against which the work is done be variable, let the path of the particle be divided into a large number (n) of small displacements, so small that the force may be considered constant during each. Let F_1, F_2, etc., be the magnitudes of the force during these small displacements, d_1, d_2, etc., the components of the displacements in the lines of action which the force has during the displacements, and v_1, v_2, etc., the velocities of the particle after the successive displacements. Then

$$F_1d_1 = \tfrac{1}{2}mV^2 - \tfrac{1}{2}mv_1^2,$$
$$F_2d_2 = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_2^2,$$
etc.
$$F_nd_n = \tfrac{1}{2}mv^2_{n-1} - \tfrac{1}{2}mv^2.$$

Hence, if W be the whole work done, we have by summation
$$W = \tfrac{1}{2}mV^2 - \tfrac{1}{2}mv^2.$$

Hence the work which can be done by a particle during a given reduction of its velocity is equal to the change produced in the product of half its mass into the square of its velocity.

If its final velocity is zero, its work-power due to its velocity is exhausted. In that case $W = \tfrac{1}{2}mV^2$. Hence the kinetic energy of a particle is equal to half the product of its mass into the square of its velocity.

The name *vis viva* was formerly given to the product mV^2.

345. *Potential Energy.*—A particle which is acted upon by such a force as that of gravitational attraction and is in a position from which it can move in the direction of the force, can in virtue of its being so acted upon do work against a second force having a component in a direction opposite to that of the first. It therefore possesses work-power or energy. Thus a heavy body in a

position from which it can fall, can do work against a force acting on it in an upward direction, water from a mill-pond, *e.g.*, against the reaction of the buckets of a mill-wheel. So also the string of a bent bow can do work against the reaction of the arrow in contact with it.

This form of energy has been called *energy of position* for an obvious reason, and *static energy* to indicate its independence of the particle's possessing a velocity. The latter term however is defective as seeming to imply that the particle possessing this form of energy must be in equilibrium (323).

A particle acted on by a force and in a position from which it can move in the direction of the force may also be recognized as possessing energy, if we note that, even if no other force be supposed to act on it, it must move in the direction of the force, gaining velocity and therefore work-power. For this reason energy of position has appeared to some writers to be simply a potential form of kinetic energy and it has been named for this reason *potential energy*. We have seen however that a particle acted on by a force and in a position from which it can move in the direction of the force, can do work without first acquiring kinetic energy; and energy of position must therefore rank, for the time at any rate, and until we acquire a deeper insight into it, as an independent form of energy. The term potential energy should not therefore be employed in the sense in which it was first proposed. We shall see however (356), that this form of energy has a very simple relation to a quantity called the potential; and to indicate this relation the term potential energy is employed.

In speaking of a particle as possessing potential energy we are taking a narrow view of the phenomenon and neglecting the third law of motion, which states that a force acting on a particle is but one aspect of a stress which acts between it and another. What we have said of the one particle applies obviously equally to the other.

Hence the potential energy belongs not to either of the particles alone but to the pair, and it is due to the stress between them. When we speak of the potential energy as possessed by the one, we are imagining the other for the moment to be immoveable, or, in other words, we are taking the position of the other as our point of reference.

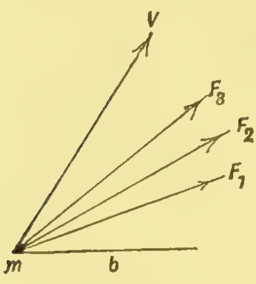
346. It follows from 345 that all forces do not confer potential energy on the particles on which they act, but those only in whose directions the particles can move. If, *e.g.*, a particle be in motion in contact with a rough surface, it will be acted upon by friction. But the direction of this force must always be opposite to that of the particle's velocity, and the particle therefore cannot move in its direction. Hence friction cannot confer potential energy on a particle. Now all natural forces may be divided into two classes, those of the one class (central forces (338), *e.g.*, gravitational attraction) depending only on the position of the particle acted upon, those of the other class (including such as friction) depending upon its velocity and having in all cases directions opposite to that of its velocity. Potential energy is thus conferred on a particle only by forces of the former class whose action depends upon the position of the particle only, and is independent of its velocity.

It should be noted that we are here making a new appeal to experience; and that the three Laws of Motion tell us nothing as to particular natural forces.

347. A particle acted upon by a central force will possess potential energy at whatever point it may be placed of the region throughout which the force acts, but the nearer it is to the centre of force the less it will have. As the force depends on the position of the particle only, the work done by the particle against the force during a motion from any point A to any more distant point B (Fig. of 338) will be equal to the work done on the

particle by the force during the opposite motion from B to A. If, during the motion from B to A the particle be acted upon by a resisting force equal at all stages to the component of the central force in the line of motion, the particle will move from B to A with uniform velocity, and will do, against the resisting force, an amount of work equal to that done on it by the central force. Thus in the position B, its potential energy is greater than it is at A by the amount of the work done by it against the central force during the motion from A to B. Also, as seen in 338, the work done during the motion from A to B is independent of the path. Hence the excess of the particle's potential energy at B over its value at A depends only on the positions of these points, and is equal to the work done against the central force during the particle's motion from A to B.

348. *The Law of Energy.*—Let a particle having any initial velocity V undergo displacement when under the action of any number of forces F_1, F_2, etc. Let R be their resultant and r the component in its direction of the displacement b of the particle; then, the acceleration of the particle being R/m, it may be shown as in 344 that $Rr = \frac{1}{2}mv^2 - \frac{1}{2}mV^2$. Now, if d_1, d_2, etc., are the components of the displacement in the directions of the forces F_1, F_2, etc., we have (342) $Rr = \Sigma Fd$. Hence $\Sigma Fd = \frac{1}{2}mv^2 - \frac{1}{2}mV^2$; *i.e.*, the algebraic sum of the quantities of work done by the acting forces is equal to the increase of the kinetic energy. If the forces be variable the above applies only to indefinitely small displacements; but it may be shown as in 344 that the same result holds for finite displacements.



We may write this result

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mV^2 + \Sigma(-Fd) = 0.$$

The quantity $\Sigma(-Fd)$ is (330) the algebraic sum of all the work done against the acting forces. Hence, in any displacement of a particle, the increase of kinetic energy together with the work done against the acting forces is zero.

If now the acting forces are all independent of the velocity of the particle, the work done against them is equal to the increase of the potential energy of the particle. Hence, in any displacement of a particle acted on by forces independent of its velocity, the sum of the increments of the kinetic and potential energies is zero, or, in other words, the sum of the potential and kinetic energies is constant. This result is the Law of the Conservation of Energy as applied to a single particle.

Forces which depend only on the position of the particle acted upon are usually called conservative forces, as being subject to the above law of conservation of energy. Those which depend on the velocity are called non-conservative forces.

If any of the acting forces are dependent upon the velocity of the particle, the work done against them does not result in the production of an equivalent amount of potential energy. In such cases, therefore, the sum of the increments of the kinetic and potential energies, and of the work done against such forces, is equal to zero. This result is the Law of Energy as applied to a single particle. The law of the conservation of energy is obviously a special case of the more general law of energy.

349. If a particle acted on by forces be in motion, its energy at any instant consists partly of kinetic, partly of potential energy. During the motion the relative amounts of these energies will in general change. In such a case its energy is said to be undergoing *transformation*. Thus

the energy of a pendulum at the extremity of its swing is wholly potential energy. In its mean position (if it be supposed that the string cannot be cut, and that the bob therefore cannot fall lower than the mean position) the energy is wholly kinetic; at intermediate positions it possesses energy of both kinds. The transformations of a particle's energy are always subject to the law of energy. Thus the sum of the kinetic and potential energies of the pendulum at any instant, together with the work done since any former instant against non-conservative forces, must be equal to the energy of the pendulum at that former instant. If the forces acting are all of the conservative class, the sum of the kinetic and potential energies of the pendulum must be the same at all instants.

350. *Examples.*

(1) Compare the amounts of the momentum and kinetic energy in (*a*) a mass of 20 lbs. having a speed of 16 ft. per sec., and (*b*) a mass of 1 oz. moving at 5,120 ft. per sec.

Ans. Momenta the same, kinetic energy of (*b*) 320 times that of (*a*).

(2) A cannon ball of 5,000 grammes is discharged with a speed of 500 metres per sec. Find the kinetic energy in (*a*) ergs, and (*b*) foot-pounds.

Ans. (*a*) $6{\cdot}25 \times 10^{12}$, (*b*) $4{\cdot}61 \times 10^{5}$, approx.

(3) A bale of goods weighing 1 cwt. is lifted 20 ft. Find the increment of its potential energy.

Ans. 2,240 ft.-pounds.

(4) A bow 1 yard long is straight when the string is just tight, but when bent has the form of a circular arc of 1 ft. 6 in. radius. The mean force exerted by the hand in bending, per unit distance through which it has moved, is equal to the weight of 10 lbs. Find the potential energy of the bow.

Ans. 483 ft.-poundals.

(5) A body is projected either (a) vertically upwards, or (b) in any direction. Show, by calculating its kinetic and potential energies after any time, that in both cases the energy of the body is the same at all points of its path. [Neglect the resistance of the air and assume g to have the the same value at all points of the path.]

(6) A meteorite falls in a straight line towards the earth from a great distance, no other heavenly body being supposed near. Show, by calculating the changes produced in its kinetic and potential energies between any two points of its path, that there is no change produced in its energy.

(7) A particle weighing 1 lb. has a simple harmonic motion with a period of 20 sec. and an amplitude of 1 ft. Find (a) its kinetic energy in its mean position, (b) its potential energy in either extreme position, (c) its kinetic energy and potential energy and their sum when at a distance of 8 inches from the mean position.

Ans. (a) $\pi^2/200$ ft.-poundals, (b) the same, (c) kinetic energy $=\pi^2/360$ ft.-poundals, potential energy $=\pi^2/450$ ft.-poundals, their sum $=\pi^2/200$ ft.-poundals.

351. *Application of the Law of Energy to Kinetic Problems.*—The law of energy is merely a particular mode of expressing the Second Law of Motion, applicable to forces of certain given kinds. It may thus be applied to the solution of kinetic problems such as those of 320. If the forces acting are all conservative, the law of the conservation of energy is applicable. If some of the forces are non-conservative, and if the work done against them cannot be determined, the law of energy cannot be applied.

352. *Examples.*

(1) What speed will the bow of 350, Ex. 4, communicate to an arrow weighing 2 oz. [Assume no work done against non-conservative forces.]

Ans. 87·9 ft. per sec.

(2) A ball weighing 5 oz. and moving with a speed of 1,000 ft. per sec. strikes a shield 2 inches thick and after piercing it moves on with a speed of 400 ft. per sec. Find the force (supposed uniform) with which the shield resisted the ball. [Assume as above.]

Ans. 787,500 poundals.

(3) Find the height (h) to which a body weighing 2 grammes and projected vertically upwards with a speed of 20 metres per sec. will have risen before its speed is reduced to 5 metres per sec., assuming the mean resistance of the air to the motion of the body per centimetre travelled to be 10 dynes.

Loss of kinetic energy $=3{,}750{,}000$ ergs, gain of potential energy $=1{,}962h$ ergs, work done against resistance $=10h$ ergs. Hence $h=1{,}901{\cdot}6$ cm.

(4) A body of mass m is projected with speed V up an inclined plane of inclination α, the coefficient of kinetic friction being μ, Find the space s traversed before the body comes to rest.

Loss of kinetic energy $=\frac{1}{2}mV^2$, gain of potential energy $=mgs\sin\alpha$, work done against friction $=\mu mgs\cos\alpha$. Hence

$$mgs\sin\alpha+\mu mgs\cos\alpha-\tfrac{1}{2}mV^2=0,$$

and $$s=V^2/[2g(\sin\alpha+\mu\cos\alpha)].$$

Hence also the acceleration is constant and equal to $g(\sin\alpha+\mu\cos\alpha)$.

(5) Find the speed v of the bob (mass $=m$) of a simple pendulum length $=l$) which has swung from its extreme position through a given angle, neglecting the resistance of the air.

Let β be the angle made with the vertical in the extreme position, θ the angle made with the vertical in the position in which the speed of the bob is to be determined. The kinetic energy gained is $\frac{1}{2}mv^2$. The vertical height through which the bob has fallen is $l\cos\theta-l\cos\beta$, and therefore the potential energy lost is $mgl(\cos\theta-\cos\beta)$. The stress in the string has done no work because the bob has had no displacement in its direction. Hence

$$\tfrac{1}{2}mv^2-mgl(\cos\theta-\cos\beta)=0,$$

and $$v^2=2gl(\cos\theta-\cos\beta).$$

The reader should solve some of the Examples of 320 and 329 by the application of the law of energy. Those of 322 cannot be solved by this method, because we as yet know too little of the forces which come into play during impact. We cannot tell whether or not they are conservative forces, nor can we calculate the work done against them.

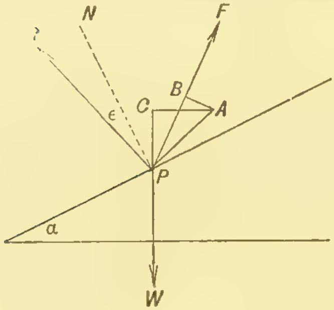
353. *Application of the Law of Energy to Static Problems.*—The law of energy may also be employed to obtain an expression for the condition of equilibrium of a particle. A particle in equilibrium must either be at rest or be moving uniformly. In any indefinitely small displacement of a particle, therefore, from a position in which it is in equilibrium, whether or not it be one which the particle actually undergoes, there can be no change of velocity, and hence no change of kinetic energy. But in any displacement the sum of the increments of potential and kinetic energies, together with the work done against non-conservative forces, must be zero. Hence in any indefinitely small displacement from a position of equilibrium the increment of the potential energy, together with the amount of work done against non-conservative forces, or, in other words, the work done against (and therefore by) all the forces acting on the particle, must be zero. With the symbols of 342, $\Sigma Fd = 0$.

This equation might have been deduced at once from that of 342, viz., $\Sigma Fd = Rr$. For, since for equilibrium $R = 0$, we have $\Sigma Fd = 0$.

A small displacement which a particle in equilibrium may be supposed to undergo is often called its *virtual displacement* or *virtual velocity*, and its product into the component of any acting force in its direction the *virtual work* or the *virtual moment* of the force. The condition of equilibrium as obtained above is then called the *Principle of Virtual Work* or of *Virtual Velocities.*

354. *Example.*

A particle of weight W is on the point of moving up an inclined plane of inclination α under a force F inclined θ to the plane, the coefficient of friction being μ. Find F in terms of W.



The inclination of the reaction R of the plane to the normal PN is $\epsilon = \tan^{-1}\mu$. As we wish to find F in terms of W, we select a displacement PA perpendicular to R. If then AB, AC be drawn perpendicular to the directions of F and W respectively, the work done by F, W, and R during the displacement are $F.PB$, $-W.PC$, and zero respectively. Hence $$F.PB - W.PC = 0,$$
and consequently $$F.\cos APB - W\cos APC = 0.$$
Now the angle APB is equal to $\theta - \epsilon$, and the angle APC to $90° - \alpha - \epsilon$. Hence $$F\cos(\theta-\epsilon) - W\sin(\alpha+\epsilon) = 0.$$
and substituting for ϵ its value $\tan^{-1}\mu$,
$$F = W.\frac{\sin\alpha + \mu\cos\alpha}{\cos\theta + \mu\sin\theta}.$$

The reader should apply this method to some of the Examples of 327 and 329.

355. *Potential.*—The region surrounding one or more centres of force (an attracting body, for example) is called a field of force. If a particle be moved from any one to any other point in such a field, work is in general done either by or against the resultant force of the field, and the amount of work so done we have seen to be independent of the path (338). If therefore some convenient point of reference be chosen, the work done in bringing a given particle, say a particle of unit mass, from any other point to the chosen point has a definite value for every point of the field. So also will the work done in carrying the given particle from the chosen point to all other points of the field. This definite value, when the given particle is one whose mass is unity, is called the

potential of the point. The magnitude of the potential of a point will depend upon the position of the point of reference, and its sign will vary according as we give the name potential to the work done during motion to or from the point of reference and by or against the force of the field. The choice of the point of reference and of the exact mode of defining potential are matters of convenience and vary with the kind of field of force under consideration.

356. The importance of the potential depends upon the following proposition:—

The rate of change of the potential per unit distance in any direction at any point of a field of force is equal to the component force in that direction with which a particle of unit mass would be acted upon if placed at that point.—Let A, B be two points in the field of force and C the chosen point of reference. Since the work done during any displacement is independent of the path of the particle, the work done in carrying unit mass from A to B is equal to the difference of the amounts of work done in carrying it from A to C and from B to C. Hence, if V_A and V_B are the potentials of A and B, the difference $V_A \sim V_B$ is equal to the work done in carrying the unit mass from A to B or from B to A. If now F is the component in AB of the mean force per unit distance acting on the particle between A and B, the work done between A and B is $F \,.\, AB$ Hence

$$F \,.\, AB = V_A \sim V_B,$$

and

$$F = (V_A \sim V_B)/AB.$$

If now B be indefinitely near A, F becomes the component force at A in the line AB, and $(V_A \sim V_B)/AB$ the rate of change of the potential at A per unit distance in the line AB. Hence the above proposition is proved.

As the value of a central force at any point of the region through which it acts is equal to the rate of

change of the potential at that point, such forces are said to be derivable from a potential.

It follows from 347 that $F\,.\,AB$ is the difference between the potential energies of unit mass at A and at B. This difference is thus equal to the difference in the values of the potential for these points. Hence the appropriateness of the term potential energy (345).

357. If at any point, $F=0$, there also the rate of change of potential must be zero. Hence *e.g.* (316, Ex. 5), at all points inside a uniform spherical shell the gravitational potential is the same.

358. *Equipotential Surfaces.*—A surface, at every point of which the potential has the same value, is called an equipotential surface. The attraction on a particle placed at any point of such a surface will be normal to the surface. For in no direction tangential to the surface is there a rate of change of potential or, consequently, a component force.

We may imagine equipotential surfaces drawn in any field of force for any values of the potential. If they be drawn for values increasing by equal amounts, which are also small, the resultant force acting at any point will be inversely proportional to the distance between successive equipotential surfaces in the neighbourhood of the point. For, if A and B are the successive equipotential surfaces, and AB the distance between them at any point, $V_A \sim V_B$ is constant, and hence (356) $F \propto 1/AB$.

359. *Lines of Force.*—A line so drawn in a field of force, that its direction at any point is also the direction of the resultant force at that point, is called a line of force. As the resultant force at a point has no component in the tangent plane of the equipotential surface passing through the point, lines of force must be normal to the equipotential surfaces they may meet.

360. *Tubes of Force.*—If from points in the boundary of any portion of an equipotential surface lines of force be drawn, the space thus marked off is called a tube of force.

361. *Gravitational Potential.*—We may consider, as of special importance, the potential in a field of force due to gravitational attraction. If such a field is due to the attraction of a single particle of mass m, the force on unit mass at unit distance (the astronomical unit of mass (315) being employed) is m. Hence (340) the work done in moving unit mass from one point at a distance r to another at a distance R is $m(1/r-1/R)$. If R is infinitely great, the work done is equal to m/r. Hence if the chosen point of reference be a point at an infinite distance from the attracting particle, the potential of a point at a distance r has the magnitude m/r. If the field is due to any number of particles of masses m_1, m_2, etc., the magnitude of the potential will be $\Sigma(m/r)$.

It is convenient to have the potential for all points of a gravitational field positive. Now gravitational force being in all cases attractive, the work done by the force of a field in moving a particle from a greater to a smaller distance from the attracting mass is always positive. Hence in this case we define the potential of a point as the work done *by* the force of the field in moving unit mass *from* a point at an infinite distance from the attracting mass, *to* the given point.

362. It follows that the component force on unit mass at a given point of a gravitational field in a given direction is equal to the rate of *increase* of the potential per unit distance in the same direction.

It follows also (347) that with the above convention, if P_A and P_B are the potential energies of unit mass at

A and B, and V_A and V_B the potentials of these points respectively,

$$V_A - V_B = -(P_A - P_B);$$

and that therefore the rate of increase of the potential with distance in a given direction is equal to the rate of decrease of the potential energy of unit mass in the same direction.

363. *Calculation of the Potential.*—The value of the quantity $\Sigma(m/r)$ for a given point may, in simple cases, be determined by elementary mathematical methods. Usually, however, the Integral Calculus is necessary to effect the summation.

364. *Examples.*

(1) Show that the potential at a given point due to particles of masses m_1, m_2, etc., situated on either a circle or a sphere whose radius is r and centre the given point, is equal to $(\Sigma m)/r$.

(2) Particles of masses 3·928, 39·28, and 392·8 kilogrammes are situated at three of the corners of a square whose side is 1 metre. Find the potential at the fourth corner.

Ans. 1·0807 C.G.S. units.

(3) Find the potential (a) at the centre of a thin circular wire of linear density ρ, and (b) at a point on a line through its centre perpendicular to its plane, distant l from all points of the wire.

Ans. (a) $2\pi\rho$, (b) $2\pi\rho r/l$.

(4) Find the potential at the centre of a circular plate of radius r and surface density ρ.

Ans. $2\pi\rho r$.

(5) Find the potential at the centre due to a sector of the plate of Ex. 4, of angle θ radians.

Ans. $r\theta\rho$.

(6) Find the potential (a) at any point inside a uniform spherical

shell of mass m and radius r, and (b) outside it at a distance d from its centre. [See 316, Exs. 5 and 6.]

Ans. (a) m/r, (b) m/d.

(7) A, a point near the earth's surface, is h feet above another such point B. Find the excess of the potential of A over that of B.

Ans. $-gh$.

365. *Integral Normal Attraction over a Surface.* If any closed surface in a field of force be divided into indefinitely small portions, the sum of the products of the areas of these portions into the normal components outwards (or inwards) of the forces exerted at them on unit mass is called the integral normal attraction over the surface (in the language of the Higher Mathematics, the surface integral of normal attraction).

The integral normal attraction over any closed surface in a gravitational field of force is equal to 4π times the mass enclosed by the surface. Let m be the mass of any

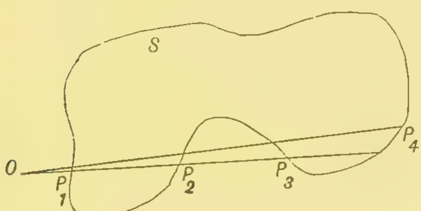


Fig 1

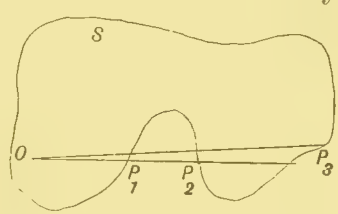


Fig 2

particle, at O, of the attracting body. Let a cone of indefinitely small solid angle meet the closed surface S at P_1, P_2, P_3, P_4, etc., marking out on it areas $A_1\, A_2\, A_3\, A_4$, etc., inclined to orthogonal sections of the cone at the angles $\theta_1, \theta_2, \theta_3, \theta_4$, etc., radians. The resultant force due to this particle at P_1, P_2, P_3, P_4, etc., is towards O and inversely proportional to $OP_1^2, OP_2^2, OP_3^2, OP_4^2$, etc. The normal components at these points are therefore proportional to $\cos\theta_1/OP_1^2$, $\cos\theta_2/OP_2^2$, etc. The orthogonal sections of the cone at P_1, P_2, etc., have areas proportional

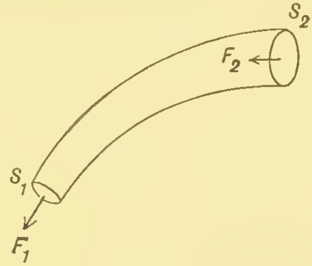
to OP_1^2, OP_2^2, OP_3^2, etc. Hence the sections inclined to them at the angles θ_1, θ_2, etc., at the same points, have, since the cone is of indefinitely small angle, areas proportional to $OP_1^2/\cos\theta_1$, $OP_2^2/\cos\theta_2$, etc. The products of the areas A_1, A_2, etc., into the respective normal components of the force over them are therefore constant.

If now the point O be outside the surface, the force at P_1 is outwards, that at P_2 inwards, that at P_3 outwards, that at P_4 inwards, and so on; and the cone must meet the surface an even number of times. Hence, if the forces at P_1, P_2, etc., be reckoned either all outwards or all inwards, as many of the above equal products are positive as negative, and their algebraic sum is consequently zero. But the whole surface may be divided into indefinitely small areas by such cones. Hence the integral normal attraction over the surface is equal to zero when the attracting mass is a particle outside it.

If the point O be inside the surface, the cone whose vertex is O will cut the surface in whatever direction it be drawn an odd number of times. Hence the sum of the products of the areas intercepted by the cone into the normal components of the attractions at them is equal to the value of the product at any one section. At P_1 the normal component of the attraction is $m\cos\theta_1/OP_1^2$. If ω is the solid angle of the cone (in solid radians), the area of the section at P_1 is $\omega \,.\, OP_1^2/\cos\theta_1$. Hence the value of the above product at P_1 is ωm, and consequently the value of the integral normal attraction is the product of m into the sum of the solid angles of all the cones with O (inside the surface) as vertex, by which the surface may be divided into small elements, which is 4π. Hence the value of the integral normal attraction when O and therefore m are inside is $4\pi m$.

Hence its value when any mass M is inside is $4\pi M$.

366. In a tube of force whose ends are indefinitely small portions of equipotential surfaces, the force perpendicular to the tubular portion of the surface vanishes, and at the end surfaces the resultant force is normal.—Let F_1 and F_2 be the resultant forces at the ends, and let s_1, s_2 be the indefinitely small areas of the ends. If then the tube contain no attracting particles, one of the two, F_1, F_2, is outwards, the other inwards, and we have, by 365, $F_1 s_1 - F_2 s_2 = 0$. Hence if F be the force at any point of a small tube of force and s its normal section at that point, $Fs =$ constant, or $F \propto 1/s$, *i.e.*, the resultant force at any point of a small tube of force is inversely proportional to its transverse section at that point.



367. If the attracting body is a uniform spherical shell or a sphere with density symmetrical with respect to the centre, the lines of force are clearly straight lines radiating from the centre, and the tubes of force are cones, right sections of which are directly proportional to the squares of their distances from the vertex of the cone or centre of the sphere. Hence the attractions exerted by a sphere such as that specified above, at external points, are inversely proportional to the squares of their distances from its centre. (Compare 316, Ex. 6.)

368. If the attracting body is a cylinder of circular section and infinite length and with density symmetrical with respect to the axis, the lines of force are clearly straight lines perpendicular to the axis of the cylinder, and the tubes of force are therefore wedges, the areas of right sections of which are directly proportional to their distances from the axis. Hence the forces at external points are inversely proportional to their distances from the axis.

369. If the attracting body is a plate of uniform thickness and infinite extent, and with the density symmetrical with respect to a plane midway between the bounding surfaces, the lines of force are clearly straight lines normal to either bounding surface, and the tubes of force therefore are cylinders of constant section. Hence the forces exerted at all external points are the same.

370. If a tube of force cut orthogonally through a plate of attracting matter of surface density ρ, and if the area of the plate inside the tube be σ, we have (365 and 366)

$$F_1 s_1 - F_2 s_2 = 4\pi\rho\sigma.$$

If the plate be indefinitely thin and the ends of the tube indefinitely near the surfaces of the plate, $s_1 = \sigma = s_2$. Hence

$$F_1 - F_2 = 4\pi\rho,$$

i.e., the attractions on unit mass on opposite sides of the plate at points indefinitely near it differ by $4\pi\rho$. As the components of these attractions, due to the plate, are clearly equal in magnitude and opposite in direction, the attraction due to the plate at either side indefinitely near the plate is thus $2\pi\rho$. (Compare 316, Ex. 1.)

371. The potential cannot have a maximum or a minimum value at a point in free space. For, if it could, it must increase or diminish respectively in all directions outwards from the point, and hence the force at all points of a small surface enclosing the given point must be outwards or inwards respectively, and the integral normal attraction must have a finite value, though the surface encloses no attracting particles.

Hence, if the potential is constant over a closed surface containing no attracting mass, it must be constant throughout the whole enclosed space. For otherwise there must be somewhere in it a point of maximum or minimum potential.

372. A field of force whose law is that of gravitation may be so mapped out by lines of force that they may indicate not only the direction, but also the magnitude of the forces acting at different parts of the field. For let s_1, s_2 be normal sections of any tube of force not enclosing any attracting matter, and F_1, F_2 the resultant forces per unit mass at these sections. Then these sections are cut through by the same number of lines of force. Let the number (n) be such that $F_1 = n/s_1$. Then, since $F_1 s_1 = F_2 s_2$, we have also $F_2 = n/s_2$. Hence, if the lines of force in a tube of force are so drawn that at any one point the quotient of their number by the normal section of the tube is equal to the force at that point, the same will be true for any other point. If therefore the lines of force of a field are so drawn that over any equipotential surface the number of lines of force per unit of area at every point is equal to the force at that point, then throughout the field the number of lines of force per unit of area normal to them at any point will be equal to the force at that point.

373. A uniform field of force is one at all points of which the resultant force has the same magnitude and direction. The tubes of force must therefore be cylinders, and the lines of force must be parallel straight lines, equal numbers of which pass through equal areas normal to their direction.

CHAPTER III.

DYNAMICS OF SIMPLE SYSTEMS OF PARTICLES.

374. For the discussion of the motion of a single particle we have found the first two laws of motion to be sufficient. If we wish, however, to discuss the motions of even only two particles which act upon one another, we have to deal with both aspects of the stress between them and must know how the stress affects both particles. The third law tells us that it affects them equally in opposite directions, producing in them equal and opposite changes of momentum in the same time. With the aid of the third law it is often possible to pass from particle to particle of a simple system, applying to each particle the equations of motion or the conditions of equilibrium for a single particle, and thus determining the motion of the whole system.

375. It is hardly necessary to point out that the law of energy also may be applied to a system of particles. For since, if the system is in motion, the increment of the potential and kinetic energies, together with the work done against non-conservative forces, during any displacement, is for each particle equal to zero, it must be equal to zero also for the whole system. And since, if the particles of the system are in equilibrium, the sum of the quantities of work done by the forces acting on each particle during any small displacement is zero, it is zero also for all together.

376. The forces acting on a system of particles may be divided into two classes, those acting between the particles of the system and bodies external to the system, called external forces, and those acting between the particles of the system themselves, called internal forces. The internal forces may be mutual attractions such as gravitational attraction, explosive forces, reactions exerted during collision, or the stresses or tensions in connecting strings. Some of these cases may be dealt with without further comment.

377. *Examples.*

(1) Two particles of masses 20 lbs. and 1 lb. respectively, initially at rest on a smooth horizontal table attract one another. After a time the heavier particle has a velocity of 10 ft. per sec. Find the velocity of the other.

Ans. 200 ft. per sec. in the opposite direction.

(2) Two attracting particles initially at rest on a smooth horizontal table are observed at a given instant to be approaching one another with a speed u, the speed of each particle being measured relatively to the other. If m and M are their masses, find their speeds v and V respectively relative to a fixed point in the table.

Ans. $v=Mu/(m+M)$, $V=mu/(m+M)$.

(3) A body having a velocity of 10 ft. per sec. in a given direction is divided by an explosion into two portions whose masses are 2 lbs. and 1 lb. respectively. Both portions move, after the explosion, in the original line of motion, and the portion of smaller mass has a velocity of 25 ft. per sec. in the original direction of motion. (*a*) Find the velocity of the other portion. (*b*) Find what it would have been had the velocity of the smaller portion been 50 instead of 25 ft. per sec. (*c*) Find the value of the explosive impulse in the latter case.

Ans. (*a*) $2\frac{1}{2}$ ft. per sec. in the given direction, (*b*) 10 ft. per sec. in the opposite direction (*c*) 40 absolute ft.-lb.-sec. units of impulse.

(4) Two particles of masses m and m' astronomical units, moving on a smooth horizontal table, attract one another according to the

gravitational law. Find the acceleration of either relative to the other when they are at a distance d.

Each is acted on by a force mm'/d^2. Hence their accelerations are m'/d^2 and m/d^2 respectively in opposite directions, and therefore the acceleration of either relative to the other is $(m+m')/d^2$. Hence each relatively to the other moves as it would if the other were fixed and had a mass $m+m'$.

378. *Collision.*—If particles come into collision, we require to know the stress between them during collision before we can determine their subsequent motion.

If the direction of the stress during the collision be known, its impulse may readily be determined, provided there be no recoil. Let m and n be the masses of two colliding particles, u and v their respective component velocities before collision and V their common component velocity after collision, in the line of action of the stress, and ϕ the impulse of the stress. As we have taken u and v both positive, one of them must be greater than the other that collision may occur, and V must be less than it and greater than the other. Let u be greater than v. Then $m(u-V)$ is the momentum lost by the one particle, $n(V-v)$ that gained by the other. Since these changes of momentum are produced by the same stress they must be equal. Hence

$$m(u-V)=\phi=n(V-v).$$

Hence also $$V=\frac{mu+nv}{m+n},$$

and $$\phi=\frac{mn(u-v)}{m+n}.$$

379. If there be recoil, the impulse Φ of the stress may be determined in terms of the value ϕ it would have if there were none. For u' and v' being the component

velocities of m and n respectively after collision, in the direction of the stress, we have

$$m(u-u')=\Phi=n(v'-v).$$

Hence
$$\frac{\Phi}{\phi}=\frac{u-u'}{u-V}=\frac{v'-v}{V-v}.$$

From the last of these equations we find

$$V=\frac{u(v-v')-v(u-u')}{v-v'-(u-u')},$$

and substituting this value of V in either of the expressions for Φ/ϕ we obtain

$$\frac{\Phi}{\phi}=1+\frac{v'-u'}{u-v}.$$

The relative velocity of the particles before collision, in the line of action of the stress, is called the velocity of approach, and their relative velocity in the same line after collision is called the velocity of recoil. With the above symbols $u-v$ is the velocity of approach, $v'-u'$ the velocity of recoil. If we call the ratio of the velocity of recoil to that of approach e, we have

$$e=(v'-u')/(u-v).$$

Hence
$$\Phi=\phi(1+e),$$

i.e., the impulse of the actual stress between two impinging particles is greater than that of the stress which would equalize their velocities in the ratio of $1+e$ to 1.

Newton found by direct experiment (380, Ex. 12) that in bodies of finite size the velocity of recoil has to the velocity of approach a constant ratio, independent of the masses and velocities of the colliding bodies and dependent only on their substance. The value of this constant ratio e therefore, for bodies of given substances, is called their *coefficient of restitution* (often, but improperly, their coefficient of elasticity).

380. *Examples.*

(1) Two particles of masses m and M moving in a straight line with velocities v and V respectively, come into collision, the stress between them during collision being in the direction of the line of motion and the coefficient of restitution being e. Find the velocities v' and V' respectively after the collision.

Clearly the particles move after collision in the same straight line as before it. Since the same stress acts on both particles during the collision, the change of momentum produced in the one is equal and opposite to the change of momentum produced in the other. Hence the sums of the momenta of the particles before and after impact are the same ; *i.e.*,

$$mv+MV=mv'+MV'.$$

V and v being both positive, the particles are moving in the same direction. If M is ahead, v must be greater than V that collision may occur. Hence the velocity of approach is $v-V$, and that of recoil $V'-v'$, and (379)

$$V'-v'=e(v-V).$$

From these equations, eliminating first V' and secondly v', we obtain

$$v'=\frac{mv+MV-eM(v-V)}{m+M},$$

$$V'=\frac{mv+MV-em(V-v)}{m+M}.$$

If the particles be moving each towards the other, one of the velocities before impact, viz. V, must be made negative in these equations.

The above result applies, as we shall see in a future section (498, Ex. 10), to the collision of spheres whose centres are before impact moving in the direction of the line joining them. The impact of spheres under this condition is said to be direct.

(2) Two particles whose masses are as 2 : 4 are moving towards one another in a straight line with speeds of 10 and 20 ft. per sec. respectively. They impinge, the stress during impact being in the

line of motion. Find the velocities after impact, the coefficient of restitution being $\frac{1}{3}$.

Ans. 50/3 and 20/3 ft. per sec. respectively in the direction of the velocity, before impact, of the particle of greater mass.

(3) Two particles of equal mass, and with coefficient of restitution equal to unity, are moving in the same straight line, and collide, the stress during collision being in the line of motion. Show that they exchange velocities.

(4) Two particles are moving in opposite directions in a straight line with equal momenta. Thy collide and do not separate after collision. Show that their kinetic energy has disappeared.

(5) Two particles of equal mass move one after the other in the same straight line, and the velocity of the hindermost is double that of the other. Show that if on colliding the stress between them is in the line of motion, their velocities after impact will be as $3-e:3+e$, e being the coefficient of restitution.

(6) A series of particles (coefficients of restitution $=1$) are moveable in a given straight line. The first of them impinges on the second, the second on the third, and so on, the stresses during impact being in the given straight line. Prove that, if their masses form a geometric progression whose common ratio is 2, their velocities after impact will form a geometric progression whose common ratio is $\frac{2}{3}$.

(7) Three particles A, B, C of different masses and materials are capable of moving in a given straight line. They are originally at rest and not in contact. A is projected with a speed V against B, which then strikes C and communicates to it a momentum M. The stresses during impact are in the given straight line. Show that, if C had been projected with the same speed V in the opposite direction, the same amount of momentum M would have been communicated to A.

(8) A particle lying at a point A on a smooth horizontal plane is driven perpendicularly against a vertical wall by the impact of another particle of equal mass moving perpendicularly to the wall

the stress during impact being in the line of motion. After rebounding from the wall at a point C, it is brought to rest by a second impact at B. Show that $BC = e \,.\, AC$, where e is the common coefficient of restitution of balls and vertical wall.

(9) Two heavy particles of equal mass (coefficient of restitution $=\frac{1}{2}$) which are in the same horizontal plane at a distance $2a$ from each other are projected with the same speed $\sqrt{ga}$ towards each other. Show that their common speed after collision will be $\frac{1}{2}\sqrt{5ga}$, the direction of the stress during impact being parallel to the line joining their initial positions.

(10) Two particles of equal mass, whose coefficient of restitution is unity, move along a smooth horizontal table with equal velocities in directions perpendicular to one another and collide at the edge, the direction of the stress during collision being that of the edge. The speed of each ball is that which it would acquire in falling from rest through a distance equal to half the height of the table. Show that the distance between the points at which the particles strike the floor is twice the height of the table.

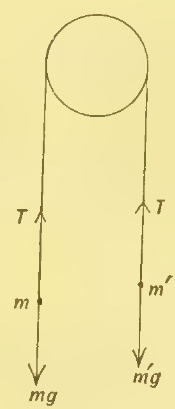
(11) A series of n particles with masses 1, e, e^2, etc., are at rest in a straight line and not in contact. To the first is given a velocity u and it impinges on the second. The second strikes the third, and so on, the stresses during all the impacts being in the line of motion. Show that the final kinetic energy of the system is $\frac{1}{2}(1-e+e^n)u^2$.

(12) Two particles of the same mass are suspended by equal strings so that they rest in contact. One of them is drawn aside through an arc whose chord is a and, being allowed to fall, drives the other up an arc whose chord is b. Show that the coefficient of restitution is $(2b-a)/a$. [It was by experiments of this kind, performed with spheres instead of particles (see 498, Ex. 10), that Newton proved the coefficient of restitution to be independent of the mass and velocity of the impinging bodies and dependent only upon their substance. The spheres were suspended from fixed points in the same horizontal plane by parallel strings of such length that the spheres rested in contact with their centres in the same horizontal plane.]

381. *Systems of Particles connected by Strings.*—We shall investigate farther on (383–397) the stresses in strings. Meantime we may assume that tense strings connecting the particles of a system are straight, except in cases in which their direction may be changed by contact with bodies or by the action of other strings knotted to them; that the stress in a straight string and in one which bends round a smooth body such as a peg, a beam, a pulley, is the same throughout; and that the stresses in strings knotted together are not in general the same.

382. *Examples.*

(1) Two particles of masses m and m' ($m > m'$) are connected by a massless inextensible string which passes over a smooth horizontal cylinder (or peg, or pulley). Find their accelerations and the tension in the string.



As the tension T is the same throughout the string, each particle is acted upon by two forces, T vertically upwards and its weight downwards. As the string is inextensible and m is greater than m', m will move downwards with the same acceleration a with which m' moves upwards. The resultant force downwards on m is $mg - T$. Hence (317) $a = (mg - T)/m$. The resultant force upwards on m' is $T - m'g$. Hence $a = (T - m'g)/m'$. Equating these values of a we obtain

$$(mg - T)/m = (T - m'g)/m'.$$

Hence
$$T = \frac{2mm'}{m + m'}g\,;$$

and substituting this value of T in either of the above expressions for a, we have
$$a = \frac{m - m'}{m + m'}g.$$

Otherwise, by applying the law of energy, thus: Let m move down and therefore m' up through a distance s, and let the initial and

final velocities be V and v respectively. Then the gain of kinetic energy is $\frac{1}{2}(m+m')(v^2-V^2)$. Equal amounts of work are done by and against T. The work done against the weight of m' is $m'gs$, that done by the weight of m is mgs. Hence the total gain of potential energy is $(m'-m)gs$. If we neglect the resistance of the air there are no other forces acting. Hence

$$\tfrac{1}{2}(m+m')(v^2-V^2)+(m'-m)gs=0,$$

and
$$v^2-V^2=2\frac{m-m'}{m+m'}gs,$$

from which it follows that the particles are moving with constant acceleration of magnitude $(m-m')g/(m+m')$.

If m and m are known and if a be observed, g may be determined. But, as no smooth bodies exist in nature and the conditions of the above ideal problem cannot therefore be realized, this mode of determining g is of no value. *Atwood's Machine*, a piece of apparatus of historic interest, is an attempt to realize as nearly as possible the above ideal arrangement. The string passes round a pulley so rough as to prevent its slipping. The axis of the pulley is mounted on "friction wheels which diminish the friction of the axis very greatly. When the particles move the pulley rotates, and the kinetic energy produced exists partially in the rotating pulley. Work is also done against friction and the resistance of the air. The complete discussion of this apparatus is therefore too complicated for us at our present stage. (See 498, Ex. 1.)

(2) At the extremity of a string which passes over a frictionless pulley moving in a vertical plane are bodies of m and 3 lbs. Initially the bodies are at rest at the same height, and 3 seconds later the former is 72 feet below the other. Show that $m=5$ lbs.

(3) Bodies of p and q grms. ($p>q$) respectively are attached to the ends of a string which passes over a pulley. At the end of each second after motion begins, 1 grm. is taken from p and added to q without jerking. Show that the motion will be reversed after $p-q+1$ seconds.

(4) Two particles of masses m and m' move on two rough inclined planes (inclinations α and α', coefficients of friction μ and μ') in a

vertical plane normal to the intersection of the inclined planes. They are connected by a string which passes over a smooth peg at the common summit of the inclined planes. Find their common acceleration, assuming m to move down its plane.

Ans. $\{m(\sin\alpha - \mu\cos\alpha) - m'(\sin\alpha' + \mu'\cos\alpha')\}g/(m+m')$.

(5) A body weighing 19 lbs. is placed at the centre of a smooth round table 6·44 ft. in diameter. It is moved by a body weighing 1 lb. at the end of a cord passing over the edge of the table. Find the time required by it to reach the edge of the table.

Ans. 2 secs.

(6) A body of 6 oz. slides down a smooth inclined plane whose height is half its length, and draws another body by means of a string along a smooth horizontal table which is level with the top of the inclined plane over which the string passes. In 5 secs. from rest it moves through 3 ft. Find the mass on the table.

Ans. 396·5 oz.

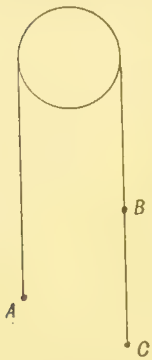
(7) A string having at one end a body of unknown mass M, and at the other a smooth massless ring, hangs over a smooth horizontal cylinder. Through the ring a second string passes, having at its ends bodies of mass m and m' $(m'>m)$. Find (a) what value M must have that m', if initially at rest, may remain at rest during the motion of the system, and (b) the acceleration of the ring.

Ans. (a) $\dfrac{4mm'}{3m-m'}$, (b) $\frac{1}{2}\cdot\dfrac{m'-m}{m}g$.

(8) Three particles A, B, C of masses m_1, m_2, m_3 $(m_1>m_2)$ are connected by strings, A to B and B to C. The string between A and B passes over a smooth horizontal cylinder. C lies on a table vertically below B and the string joining B and C is slack. At a given instant A and B begin to move from rest, and after t sec. the string between B and C tightens. Determine the subsequent motion.

The acceleration with which A and B move while BC is slack is Ex. 1, $(m_1-m_2)g/(m_1+m_2)$. Hence their velocity at the instant at which BC becomes tight is $(m_1-m_2)gt/(m_1+m_2)$. Call this v. Let u be the common speed of A, B, and C immediately after the tightening of BC. Then C's momentum upwards has suddenly increased

by m_3u. It has therefore been acted upon by a short-lived stress of impulse m_3u upwards. A's momentum in an upward direction has changed from $-m_1v$ to $-m_1u$. Hence it has been acted upon by a stress of impulse $m_1(v-u)$ upwards. Both these stresses have acted on B—the latter upwards, the former downwards. Hence the impulse of the resultant upward stress on B is $m_1(v-u)-m_3u$. Now B's momentum upwards has changed from m_2v to m_2u. Hence (319)



$$m_1(v-u)-m_3u=m_2(u-v),$$

and
$$u=\frac{m_1+m_2}{m_1+m_2+m_3}v=\frac{m_1-m_2}{m_1+m_2+m_3}gt.$$

The acceleration with which A, B, and C move after BC becomes tight may be shown (as in Ex. 1) to be

$$\frac{m_1-m_2-m_3}{m_1+m_2+m_3}g.$$

Hence the subsequent motion is determined. If $m_2+m_3>m_1$, the acceleration is negative, the velocity will gradually diminish from u to zero, and the direction of motion will then be reversed.

We cannot apply the law of energy to a problem such as the above, because we do not know what non-conservative forces may be acting, and cannot therefore determine the work done against them.

(9) Three bodies A, B, C of equal mass are connected by strings A to B and B to C. A and B are placed close together on a smooth horizontal table, and C hangs over the edge. The string AB is 3 ft. long, and B is 3·5 ft. from the edge. Find the velocity of A (a) when it begins to move, and (b) when B arrives at the edge.

Ans. (a) $2\sqrt{g/3}$, (b) $\sqrt{5g/3}$.

(10) Two particles A and B of equal mass are connected by a string which passes over a smooth horizontal cylinder. While moving with a speed v (A moving downwards) a third particle, C, of the same mass and at rest, is suddenly attached to the string between A and the cylinder. Find (a) the common speed of A and

C immediately after C's attachment, (b) the time after which the string BC again becomes tight, (c) the common speed of A and C and the speed of B just before the string BC tightens, and (d) the common speed of all three just after the string tightens.

Ans. (a) $v/2$, (b) $v/(2g)$, (c) v, $v/2$; (d) $5v/6$.

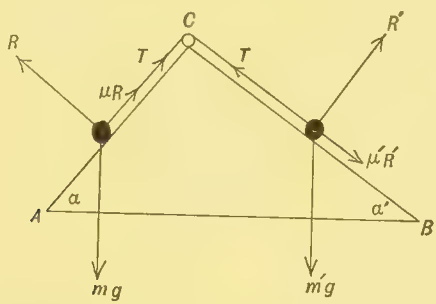
(11) In Ex. 4, find the ratio of the masses m and m' that there may be equilibrium with m on the point of moving down its plane.

Equating to zero the value of the acceleration found above (Ex. 4), we have

$$m(\sin\alpha - \mu\cos\alpha) - m'(\sin\alpha' + \mu'\cos\alpha') = 0,$$

and
$$\frac{m}{m'} = \frac{\sin\alpha' + \mu'\cos\alpha'}{\sin\alpha - \mu\cos\alpha}.$$

Otherwise thus: The acting forces are represented in the figure



which will be understood without explanation. Resolving forces parallel to the planes we obtain

$$T - mg\sin\alpha + \mu R = 0,$$

$$T - m'g\sin\alpha' - \mu' R' = 0.$$

Resolving in directions perpendicular to the planes we obtain

$$R - mg\cos\alpha = 0,$$

$$R' - m'g\cos\alpha' = 0.$$

Hence, substituting for R and R' in the first pair of equations their values as given in the second pair and eliminating T, we obtain

$$\frac{m}{m'} = \frac{\sin\alpha' + \mu'\cos\alpha'}{\sin\alpha - \mu\cos\alpha}.$$

Otherwise thus: Let the particles move a short distance s, m down the plane CA and m' up the plane CB. Then, equating to zero the algebraic sum of the amounts of work done by the various forces on both particles during the displacement, we have

$$mgs \sin \alpha - m'gs \sin \alpha' - \mu Rs - \mu' R's = 0;$$

and substituting for R and R' their values $mg \cos \alpha$ and $m'g \cos \alpha'$ respectively, we obtain the same result as above.

(12) Two particles A and B of weights W and W' are connected by a string. A rests on a rough inclined plane (inclination $= \alpha$, coefficient of friction $= \mu$) over whose smooth summit the string passes; and B hangs freely. Find the ratio of W' to W that there may be equilibrium with A on the point of moving up the plane.

Ans. $\sin \alpha + \mu \cos \alpha$.

(13) A string fastened by one end at a fixed point A passes through a fixed smooth ring at B, AB being horizontal, and is pulled by a force at its other end. Between A and B a body of weight W is hung by a smooth ring moveable on the string. How near to AB will it be possible to raise this ring by pulling at the string, if the string can bear a tension equal to $2W$ only.

Ans. $AB/60^{\frac{1}{2}}$.

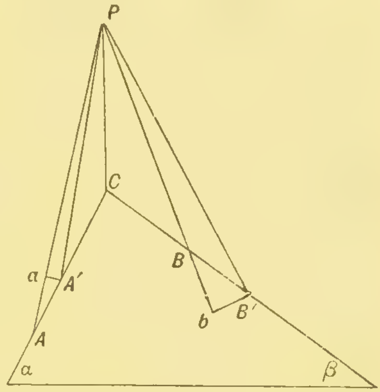
(14) A rough parabolic wire is placed with vertex upwards and axis vertical. A small ring of weight W moving on a wire is supported at one extremity of the latus rectum by a body of weight W' attached to a string passing over a smooth peg at the focus. Find the coefficient of friction.

Ans. $(W - W')/(W + W')$.

(15) Two small smooth rings sliding on a circular wire in a vertical plane are connected by a string which passes through a small smooth fixed ring at the highest point of the circle. Show that if the masses of the moveable rings are inversely proportional to the adjacent segments of the string, there will be equilibrium.

(16) Two particles A and B (masses m and m') rest upon smooth inclined planes of inclinations α and β respectively. They are con-

nected by a string (length $=l$) which passes over a smooth peg P, vertically over the common summit C of the two planes, and distant h feet from it. Find the inclinations θ and ϕ of AP and BP to the planes.



Let A be displaced up its plane through an indefinitely small distance AA'. Then B will move through a small distance BB' down its plane. From A' and B' draw $A'a$ and $B'b$ perpendicular to PA and PB produced, respectively. Since the angles APA' and BPB' are indefinitely small, $A'P=aP$ and $B'P=bP$. Hence

$$aP+Pb=A'P+PB'=AP+PB,$$

and therefore $Aa=Bb$. The angles APA' and BPB' being indefinitely small, PA and PB may be considered the directions of the tension T in the string during the displacement. Hence the amounts of work done by the tension are $T.Aa$ and $-T.Bb$, which are equal and of opposite sign. No work is done by the normal reactions. The amount of work done by the weight of A is $-mg\sin\alpha . AA'$, which is equal to $-mg . Aa . \sin\alpha/\cos\theta$. That done by the weight of B is $m'g . Bb . \sin\beta/\cos\phi$. Hence

$$m'g . Bb . \sin\beta/\cos\phi - mg . Aa . \sin\alpha/\cos\theta = 0,$$

and
$$\frac{m'}{m}=\frac{\sin\alpha}{\cos\theta}\cdot\frac{\cos\phi}{\sin\beta}.$$

We have moreover

$$AP : PC = \sin ACP : \sin PAC$$
$$= \cos\alpha : \sin\theta,$$

and
$$BP : PC = \cos\beta : \sin\phi.$$

Hence
$$AP+PB=PC\left(\frac{\cos\alpha}{\sin\theta}+\frac{\cos\beta}{\sin\phi}\right)$$

or
$$l=h\left(\frac{\cos\alpha}{\sin\theta}+\frac{\cos\beta}{\sin\phi}\right).$$

This equation, with that obtained above, are sufficient to determine θ and ϕ when there is equilibrium.

The reader should solve the problem also by applying the conditions of equilibrium in other ways.

(17) Two particles, of masses m and m', connected by a string, rest upon the edge of a smooth vertical circular disc. Find the position of equilibrium and the tension T in the string.

Ans. If α is the angle subtended at the centre by the string and β the angle subtended by the portion between m and the highest point,

$$\beta=\tan^{-1}\frac{m'\sin\alpha}{m+m'\cos\alpha},\quad T=\frac{mm'g\sin\alpha}{(m^2+m'^2+2mm'\cos\alpha)^{\frac{1}{2}}}.$$

(18) Three smooth tacks A, B, C are driven into a vertical wall, B and C being on the same level. A string, to whose ends bodies of equal weight w are attached, is hung over the three tacks. Find the forces exerted by them on the string when there is equilibrium.

Ans. $w2^{\frac{1}{2}}(1+\cos A)^{\frac{1}{2}}$, $w2^{\frac{1}{2}}(1-\sin B)^{\frac{1}{2}}$, $w2^{\frac{1}{2}}(1-\sin C)^{\frac{1}{2}}$ respectively.

(19) Two rough bodies rest on an inclined plane and are connected by a string which is parallel to the plane, W and W' being the weights of the bodies, and μ, μ' their coefficients of friction, and the rougher body having the higher position. Find the greatest inclination of the plane consistent with equilibrium.

Ans. $\tan^{-1}[(\mu W+\mu' W')/(W+W')]$.

(20) Two bodies weighing A and B lbs. respectively are connected by a string and placed on a rough horizontal table (coefficient of friction $=\mu$). A force P, which is less than $\mu(A+B)$ but greater than $\mu\sqrt{A^2+B^2}$, is applied to A in the direction BA, and its direction is gradually turned round through an angle θ in a horizontal plane. Show that both A and B will slip when

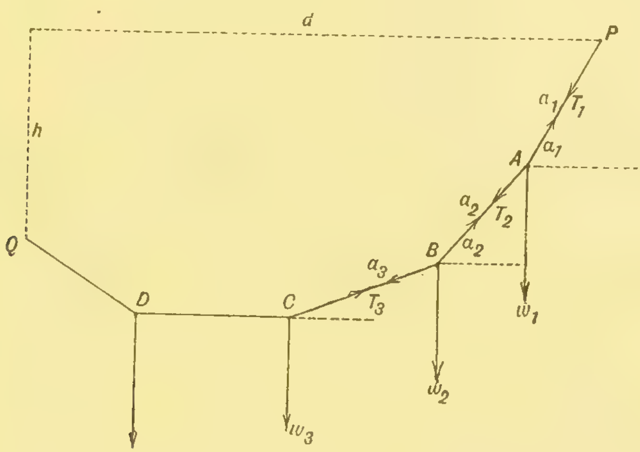
$$\theta=\cos^{-1}\frac{\mu^2(B^2-A^2)+P^2}{2\mu BP}.$$

Show also that, if P is less than $\mu\sqrt{A^2+B^2}$ but greater than μA, it will cause A only to slip, and that A will slip when $\theta=\sin^{-1}(\mu A/P)$.

(21) Particles A, B, etc., n in number, of weights w_1, w_2, etc., are connected to one another and to two fixed points P and Q, whose

horizontal distance is d and vertical distance h, by weightless strings, P being connected to A, A to B, and so on, and the last to Q. The string connecting P and A has the length a_1, that connecting A and B the length a_2, and so on. Find the tensions T_1, T_2, etc., in these strings and their inclinations α_1, α_2, etc., to the horizon, when the particles are in equilibrium.

Each particle is acted upon by three forces, its weight and the



tensions in the strings attached to it. Since w_1, w_2, etc., are all vertical, and since T_1 and T_2 are in the same plane as w_1, T_2 and T_3 in the same plane as w_2, and so on, the whole system must be in the vertical plane through P, Q. A is in equilibrium under the forces T_1, T_2, and w_1. Hence, resolving horizontally and vertically, we get

$$T_1\cos\alpha_1 - T_2\cos\alpha_2 = 0, \text{ and } T_1\sin\alpha_1 - T_2\sin\alpha_2 - w_1 = 0.$$

Similar equations may be obtained for each particle—in all $2n$ equations. Moreover

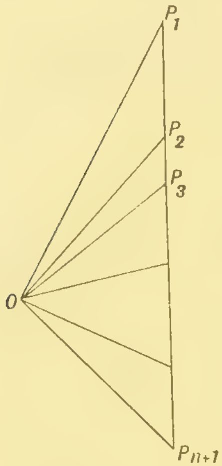
$$d = a_1\cos\alpha_1 + a_2\cos\alpha_2 + \text{etc.} + a_{n+1}\cos\alpha_{n+1},$$

and

$$h = a_1\sin\alpha_1 + a_2\sin\alpha_2 + \text{etc.} + a_{n+1}\sin\alpha_{n+1}.$$

We have therefore $2n+2$ equations involving $2n+2$ unknown quantities, viz., $n+1$ tensions and $n+1$ inclinations. The inclinations of the strings being determined and their lengths given, the positions of the particles are known.

(22) Particles A, B, C, etc., n in number, are connected by weightless strings A to B, B to C, etc., and the n^{th} to a fixed point Q. A force of given magnitude T_1 is applied to A through the string PA. Find the weights of the particles that the strings PA, AB, etc., may have given inclinations α_1, α_2, etc., to the horizon.



From any point O draw OP_1, OP_2, etc., with inclinations α_1, α_2, etc., to the horizon. These lines have therefore the same directions as the strings PA, AB, etc. Draw a vertical line meeting OP_1, OP_2, etc., in P_1, P_2, etc. If T_2, T_3, etc., are the tensions in AB, BC, etc., the particle A is acted upon by three forces, T_1, T_2, and w_1. These are represented in direction by the lines OP_1, P_2O, and P_1P_2 respectively. Hence (325, d) they are also represented by these lines in magnitude. Similarly the forces acting on B, viz., T_2, T_3, and w_2, are represented in direction and therefore also in magnitude by OP_2, P_3O, and P_2P_3 respectively. Thus it may be shown that P_1P_2, P_2P_3, P_3P_4, etc., P_nP_{n+1} represent the weights w_1, w_2, etc., on the same scale as that on which OP_1 represents T_1. Hence the values of w_1, w_2, etc., may be determined by carefully drawing the diagram (called a *force diagram*) and measuring the lengths of P_1P_2, P_2P_3, etc. For this reason the above method is called a *graphic method*. It is of great practical value for the rapid solution of engineering problems.

(23) Particles A, B, C, etc., are connected together and to two fixed points, as in Ex. 21, and are in equilibrium, their masses m_1, m_2, etc., and the inclinations of the strings α_1, α_2, etc., being known. Any one of the strings is cut, say BC. Find the tensions t_1, t_2 in PA, AB respectively immediately afterwards. [These tensions are called *initial tensions*, because they are the tensions when C, B begin to move.]

A moves, after the cutting of the string, in a circle about P. At the beginning of its motion its speed is zero, and hence the component of its acceleration normal to its path (*i.e.*, in the direction AP) is zero. Its acceleration is therefore initially wholly tangential

to its path (*i.e.*, in a direction perpendicular to AP). A's acceleration in the direction AP being zero, the sum of the components in that direction of the forces acting on A is zero also. Hence

$$t_1 - t_2 \cos(\alpha_1 - \alpha_2) - m_1 g \sin \alpha_1 = 0.$$

A's acceleration in a direction perpendicular to AP is the quotient by its mass of the sum of the component forces acting on it in this direction, and is therefore

$$[m_1 g \cos \alpha_1 - t_2 \sin(\alpha_1 - \alpha_2)]/m_1.$$

B also after the cutting of the string moves in a circle about A, and as above it may be shown that it has no acceleration relative to A in the direction BA. Hence its acceleration in this direction is equal to A's component acceleration in the same direction, and is therefore equal to

$$\frac{m_1 g \cos \alpha_1 - t_2 \sin(\alpha_1 - \alpha_2)}{m_1} \cdot \sin(\alpha_1 - \alpha_2).$$

But it is also equal to $(t_2 - m_2 g \sin \alpha_2)/m_2$. Hence

$$\frac{t_2}{m_2} - g \sin \alpha_2 = g \cos \alpha_1 \sin(\alpha_1 - \alpha_2) - \frac{t_2}{m_1} \sin^2(\alpha_1 - \alpha_2).$$

We have therefore two equations containing no other unknown quantities than t_1 and t_2, which therefore may be determined. The instantaneous changes of tension on cutting BC are of course $T_1 - t_1$ and $T_2 - t_2$, where T_1 and T_2 are the tensions before cutting as determined in Ex. 21.

(24) A particle is connected by two equal strings to two points in the same horizontal line and is in equilibrium. Show that, according as the inclination of the strings is less or greater than a right angle, will the tension of either string be instantaneously increased or diminished by cutting the other.

CHAPTER IV.

DYNAMICS OF FLEXIBLE INEXTENSIBLE STRINGS.

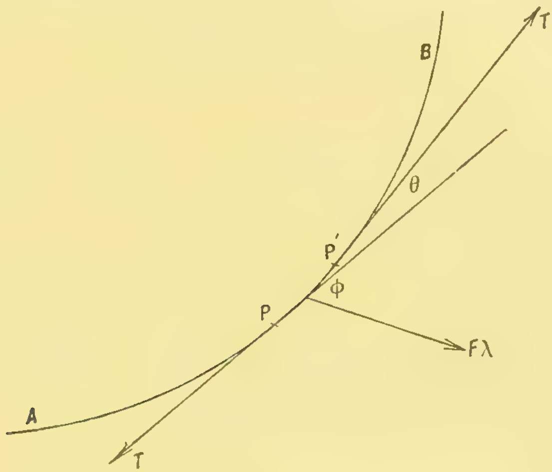
383. A string or cord or chain may be considered to be a series or row of particles or elements placed end to end. It may thus be regarded as a system of particles less simple than those of Chapter III., but more simple than those of subsequent chapters.

A perfectly flexible string is one which is capable of being bent without the exertion of any finite force. An inextensible string is one whose length is constant. Flexible and inextensible strings are ideal. Real strings all require force to bend them and can be elongated. In many cases however the forces required to bend real strings are so slight and the elongations under the acting forces so small that they may be considered to be practically perfectly flexible and inextensible.

Since such a string may be bent at any point without the exertion of any finite force, the internal forces acting at that point can have no component normal to the direction of the string. For, otherwise, this component would have to be overcome in bending the string and a finite force would be necessary. Hence the stress in a flexible string has at any point the direction of the string at the point.

We restrict our attention to the simple case in which the string itself and the external forces acting on it are in the same plane.

384. *Equations of Motion.*—Let AB be a tense string of which PP' is any element. Let the stresses in the



string at P, P' be T, T'. Then the element PP' is acted upon at its end-points by forces T, T' tangential to the string at P, P' respectively. Let it also be acted upon by some external force whose magnitude we may indicate by the product $F\lambda$, where λ is the length of the element PP' and F consequently the magnitude of the external force acting on the string per unit length of the string. Let the lines of action of T and T' be inclined at the angle θ, those of T and $F\lambda$ at the angle ϕ. Also, let σ be the linear density of the string at PP' and a_t and a_n the components of the acceleration of the element in directions tangential and normal to the string at P. Then, resolving tangentially and normally, we have, as the equations of motion of the element (317),

$$T' \cos \theta - T + F\lambda \cos \phi = \sigma\lambda a_t,$$
$$T' \sin \theta - F\lambda \sin \phi = \sigma\lambda a_n.$$

385. *Conditions of Equilibrium.*—Putting $a_t = a_n = 0$ in the equations of motion we obtain those of equilibrium, viz.,

$$T' \cos\theta - T + F\lambda \cos\phi = 0,$$

$$T' \sin\theta - F\lambda \sin\phi = 0.$$

386. The above equations hold for every element of the string. The results which may be deduced from them will vary with the nature of the external forces.

387. (1) *No External Forces.*—If there are no external forces, $F=0$. Hence the equations of motion become

$$T' \cos\theta - T = \sigma\lambda a_t,$$

$$T' \sin\theta = \sigma\lambda a_n.$$

Ultimately, when P' is very near P, θ is indefinitely small, and consequently $\cos\theta = 1$ and $\sin\theta = \theta$. Also, ultimately T' is indefinitely nearly equal to T, and $\lambda/\theta = \rho$ the radius of curvature at P. Hence the above equations become

$$(T' - T)/\lambda = \sigma a_t,$$

$$T/\rho = \sigma a_n.$$

If we are dealing with ideal massless strings or with real strings whose mass may be neglected, we have $\sigma = 0$. Hence $T = T'$ and $1/\rho = 0$; *i.e.*, the tension is the same throughout the string and the string has no curvature.

If the various elements of the string are in equilibrium, we have $a_t = a_n = 0$, and therefore in this case, even though the string be not massless, we have also $T = T'$ and $1/\rho = 0$. The string is straight and the tension is the same throughout.

388. *Examples.*

(1) An endless string of uniform linear density σ, but without weight, is moving so that the velocity of each element has a constant magnitude V, and a direction continually tangential to the

string. Show that the tension is the same throughout, and find it.

As the tangential velocity is constant $a_t = 0$. Hence $T' - T = 0$. If ρ be the radius of curvature at any point, $a_n = V^2/\rho$. Hence $T = \sigma V^2$.

(2) An endless circular string of radius r and of uniform linear density σ, but without weight, is spinning in its own plane about its centre with the angular velocity ω. Find its tension.

Ans. $\sigma\omega^2 r^2$.

389. (2) *The External Forces acting at Isolated Points*, as, *e.g.*, when $F\lambda$ is the stress in a second string knotted at PP', or the force exerted by a small peg in contact with AB at PP'. In this case, if there is equilibrium, PP' will be in equilibrium under the three forces T, T', $F\lambda$. Hence T and T' must have such directions and magnitudes that the resultant of the three may be zero. In general therefore T and T' will have different values. Only in the case in which they are equally inclined to $F\lambda$ will they be equal. The portions of the string between the isolated points at which the forces act are portions on which no external forces act. To these portions therefore the results of 387 apply. (See 382, Exs. 21–24.)

390. (3) *The External Forces continuously applied throughout the String, i.e.*, so applied that the forces acting on contiguous equal elements have indefinitely nearly the same magnitude and direction. In this case the curvature of the string is clearly continuous, θ therefore indefinitely small, and $\lambda/\theta = \rho$. Hence the equations of motion become

$$(T' - T)/\lambda + F\cos\phi = \sigma a_t,$$
$$T/\rho - F\sin\phi = \sigma a_n,$$

and the conditions of equilibrium

$$(T' - T)/\lambda + F\cos\phi = 0,$$
$$T/\rho - F\sin\phi = 0.$$

Hence, when there is equilibrium, the rate of change of the stress in the string at any given point, with respect to its distance measured along the string from a fixed point in the string, is equal to the tangential component of the external force per unit of length, at the given point; and the curvature of the string at any point is equal to the ratio of the normal component of the external force (per unit length of the string) to the stress at that point.

As instances of external forces continuously applied, we may take the reactions of continuously curved surfaces on strings wound round them, and the weights of heavy strings.

391. (*a*) *The External Force being the Reaction of a Continuously Curved Surface.*—First, let the surface be a smooth one over which the string is stretched. Then, as we are supposing the string to have no weight (and in many cases the weight is so small relatively to the stress that it may be neglected), each element of the string is acted upon by three forces only, viz., the reaction of the surface, $F\lambda$, normal to the surface, and the tensions, T, T', whose directions are those of consecutive tangents to the string. Hence in the special case to which we restrict ourselves (383) the osculating plane (41) of the string at any point is normal to the curved surface, and the form of the string is that of what is called a geodetic line on the surface. Since $F\lambda$ is normal to the surface and therefore to the string, $\phi=\pi/2$. Hence the equations of motion (390) become

$$(T'-T)/\lambda=\sigma a_t,$$

$$T/\rho-F=\sigma a_n.$$

If σ is so small that it may be neglected, we thus have $T'-T=0$ and $T/\rho=F$, or, in words, the tension in the string is the same throughout, and the reaction of the surface per unit length of the string is equal to the product of the tension into the curvature.

If there is equilibrium, T, T' and $F\lambda$ must in all cases be in the same plane, and the form of the string must therefore in all cases be that of a geodetic line. Since for equilibrium $a_t = a_n = 0$, we have, even if σ cannot be neglected, $T' - T = 0$ and $T/\rho = F$. If the curved surface be that of an indefinitely small peg, its reaction may be considered a single force, and its direction will be equally inclined to the directions of the string on each side of the peg. (See 381 and 382.)

392. Secondly, let the surface over which the string is stretched be a rough one. As before, each element is acted on by three forces, the two tensions and the reaction of the surface; but the last will not in general be normal to the surface. The conditions of the special case to which we restrict ourselves (383) may be realized, however, if the string tend to slip in its own direction. If in this case we resolve F into its normal component R and its tangential component μR, and if we suppose that the string tends to slip in the direction of T', we have (390), as equations of motion, since $F \sin\phi = R$ and $F\cos\phi = -\mu R$,

$$(T' - T)/\lambda - \mu R = \sigma a_t,$$

$$T/\rho - R = \sigma a_n,$$

where μ is the coefficient of kinetic friction.

If there is equilibrium therefore, we have

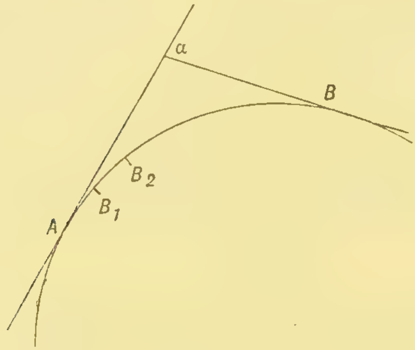
$$(T' - T)/\lambda - \mu R = 0,$$

$$T/\rho - R = 0,$$

where μ may have any value up to that of the coefficient of static friction, which it will have when the string is on the point of slipping. Eliminating R, and noting that $1/\rho = \theta/\lambda$, we obtain

$$T' = T(1 + \mu\theta).$$

Let A, B be any two points of the string in contact with the rough surface, the stress in the string at A being T_0 and the direction of the tendency to slip being from A towards B. Let a be the angle between the tangents at A and B. Divide AB into an indefinitely great number (n) of elements AB_1, B_1B_2, etc., of such length that in each case the tangents at the ends are inclined at the angle θ. Then $\theta = a/n$. Let the stress in the string have at B_1, B_2, etc. the values T_1, T_2, etc., and at B the value T. Then



$$T_1 = T_0\left(1 + \frac{\mu a}{n}\right),$$

$$T_2 = T_1\left(1 + \frac{\mu a}{n}\right) = T_0\left(1 + \frac{\mu a}{n}\right)^2,$$

etc.,

$$T = T_0\left(1 + \frac{\mu a}{n}\right)^n = T_0 e^{\mu a},$$

where e is the base of Napier's logarithms (2·71828...).* Hence, as a increases in an arithmetical ratio, T increases in a geometrical ratio.

393. *Examples.*

(1) A rope attached to a ship is wrapped three times round a rough cylindrical post (coefficient of friction $=0·5$). If a man pull at one end of the string with a force of 50 pounds-weight, what force must be exerted by the ship at the other end to bring the string to the point of slipping in its direction. [Assume the osculating plane of the string to be everywhere normal to the sur-

* See chapter on Exponential and Logarithmic Series in any Algebra.

face. In other words, assume the thickness of the string to be negligible.]

Ans. About 619,500 pounds-weight.

(2) A string hanging over a rough horizontal cylinder is on the point of slipping, with 10 lbs. hanging at one end and 1 lb. at the other. Find the coefficient of friction between the cylinder and string.

Ans. 0·73....

(3) Over two parallel horizontal rough cylinders of equal radius, whose axes are in the same horizontal plane, a string hangs in a plane perpendicular to their axes. The coefficient of friction is 0·5. Find the masses of the particles which must be attached to the ends of the string, that a particle weighing 1 lb. and hanging from a smooth ring which slides on the string between the cylinders may be in equilibrium and on the point of moving upwards when the portions of the string on each side of the ring are inclined 60° to the vertical.

Ans. 2·85 lbs. nearly.

394. (*b*) *The External Force being the Weight of the String.*—The weight of unit length of the string being σg, we have $F=\sigma g$. The equations of motion (390) thus become

$$(T'-T)/\lambda+\sigma g\cos\phi=\sigma a_t,$$
$$T/\rho-\sigma g\sin\phi=\sigma a_n;$$

and those of equilibrium

$$(T'-T)/\lambda+\sigma g\cos\phi=0,$$
$$T/\rho-\sigma g\sin\phi=0.$$

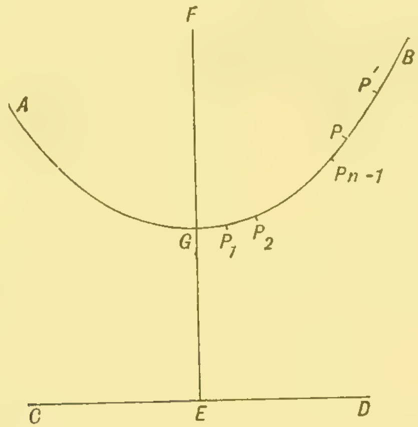
We may consider two special cases.

395. *Case I.—That of a string hanging vertically in equilibrium.* If P' be a point above P, $\phi=\pi$, since $F\lambda$ is now directed vertically downwards. Hence

$$T'-T=\lambda\sigma g,$$
$$T/\rho=0.$$

The first equation asserts that the stress increases from below upwards, and that the difference of stress between two points indefinitely near is equal to the weight of the intervening portion of string. Hence, by summation, the difference between the stresses at two points whose distance is finite, is equal to the weight of the portion of string between them. The second equation asserts that the string is straight.

396. *Case II.—That of a uniform string hanging in equilibrium with its end points fixed.*—Let A, B be



the fixed points, PP' an element of the string. From 394 we have

$$T' - T = -\sigma g \lambda \cos\phi = \sigma g \lambda \cos(\pi - \phi).$$

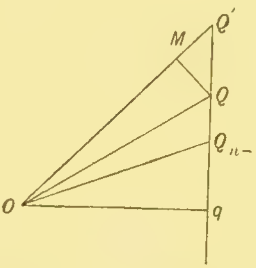
Ultimately, when PP', and therefore θ, are indefinitely small, ϕ becomes (384) the inclination of PP' to a vertical line from P drawn downwards. Hence $\lambda \cos(\pi - \phi)$ is the projection of PP' on a vertical line. Let the distances of P, P' from any horizontal line CD be y, y' respectively. Then $\lambda \cos(\pi - \phi) = y' - y$, and

$$(T' - T)/(\sigma g) = y' - y.$$

Now σg is the weight of unit length of the string. If therefore we adopt this weight as our unit of force, we have

$$T' - T = y' - y.$$

Let CD be so chosen that its distance a from G, the lowest point of the string, may be numerically equal to the tension T_0 at G. Then $T_0 = a$; and since, as we go along the string from G to P', the increment of T is equal to that of y, we have at P, $T = y$, *i.e.*, the tension at any point P is numerically equal to the distance of P from the line CD.



From any point O draw OQ' and OQ representing T' and T, and therefore proportional to y' and y respectively. Then, since T, T' and the weight of PP' are in equilibrium, and since QO and OQ' represent the tensions acting on PP', $Q'Q$ will represent the weight, and be proportional to the length, of PP'. Similarly, if OQ_{n-1} represent T_{n-1}, the tension at P_{n-1}, QQ_{n-1} will represent the weight of PP_{n-1}, and will consequently be in the same straight line as $Q'Q$ and proportional to the length of PP_{n-1}. Similarly, the portion of the line $Q'Q$ produced which is intercepted by lines representing the tensions at the ends of any element will be proportional to the length of that element; and consequently, by summation, the portion of $Q'Q$ produced which is intercepted by lines representing the tensions at any two points of the string will be proportional to the length of the string between those points. If s, s' represent the length of arc between G and P, G and P', we have thus $Q'Q$ proportional to $s' - s$. From O draw Oq perpendicular to $Q'Q$ produced. Then clearly Oq represents the tension T_0 at G and is consequently proportional to a, and Qq is proportional to s.

From Q draw QM perpendicular to OQ'. Ultimately OQ is equal to OM, and hence MQ' to $OQ'-OQ$. Also, since $OQ'Q$ is the inclination of PP' to the vertical, and $Q'Q$ is proportional to PP', MQ is proportional to the horizontal projection of PP'. Let the distances of P, P', etc., from a vertical line EF, through G, be called x, x', etc. Then MQ is proportional to $x'-x$.

Since the angle $OQ'Q$ is ultimately equal to the angle OQq, the triangle $MQ'Q$ is similar to the triangle qQO. Hence

$$Q'Q/OQ = Q'M/Qq = MQ/Oq,$$

or

$$(s'-s)/y = (y'-y)/s = (x'-x)/a.$$

Hence

$$\frac{s'-s+y'-y}{s+y} = \frac{x'-x}{a},$$

and

$$s'+y' = (s+y)\left(1+\frac{x'-x}{a}\right).$$

Let points P_1, P_2, etc., P_{n-1} be so chosen between G and P that the projections of the elements P_1G, P_2P_1, etc., on CD may be equal. Then the projection of each element on CD is x/n, n being an indefinitely large number; and, the values of s and y for the point G being zero and a respectively, we have, if s_1, y_1, s_2, y_2, etc., are the values of s and y at P_1, P_2, etc.,

$$s_1+y_1 = a\left(1+\frac{x}{na}\right),$$

$$s_2+y_2 = (s_1+y_1)\left(1+\frac{x}{na}\right) = a\left(1+\frac{x}{na}\right)^2,$$

etc.,

$$s+y = a\left(1+\frac{x}{na}\right)^n = ae^{\frac{x}{a}},$$

where e is the base of Napier's logarithms. (See 392.)

By a similar process, it may be shown that

$$s-y=-ae^{-\frac{x}{a}}.$$

Hence $$2s=a\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right),$$

and $$2y=a\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right).$$

The first of these equations expresses the relation between the weight of string (s), and the horizontal distance (x), between any given point of the string and its lowest point, and the tension (a) at the lowest point. The second expresses the relation between the distances of any point of the string from CD and EF, and enables us therefore to draw the curve in which it hangs. This curve is called the *common catenary.*

Case III.—That of a string of non-uniform linear density, hanging in equilibrium with its end points fixed. From 394, we have

$$T=\rho\sigma g \sin\phi$$

as the tension at any point P. The portion of the string GP (396), being in equilibrium under the tension T, acting at P, the horizontal tension T_0 acting at G, and its weight W,

$$T\cos\left(\phi-\frac{\pi}{2}\right)=T_0, \quad \text{and} \quad T\cos(\pi-\phi)=W.$$

Hence $$T\sin\phi=T_0 \quad \text{and} \quad T\cos\phi=-W.$$

Hence also $$T_0=-W\tan\phi,$$

and $$T_0=\rho\sigma g\sin^2\phi.$$

If we wish to find the linear density at all points of a string for which the weight of GB is given, that it may hang in a given curve, the value of ϕ at B being known and of W for GB, T_0 may be found from the second last equation; and ρ and ϕ being known for every point of

the string, the linear density may then be found from the last equation.

397. *Examples.*

(1) A body weighing 7 lbs. is suspended from a fixed point by means of a uniform string 12 inches long weighing 18 oz. Find the stress in the string at its middle point and at its upper and lower ends ($g=32$).

Ans. 242, 260, and 224 poundals respectively.

(2) A heavy uniform chain, whose extremities are A and B, can move freely over a small smooth pulley placed at the highest point of a smooth inclined plane. Show that the chain will be in equilibrium if the line AB is horizontal.

(3) Show that the horizontal component of the tension at any point of a uniform string hanging in equilibrium from two fixed points is equal to the tension at the lowest point, and that the vertical component is equal to the weight of the portion of the string between the given point and the lowest point.

(4) Show that at any point of a uniform string which is hanging in equilibrium with two points fixed, its inclination to the horizon is the angle whose tangent is the ratio of the weight of the portion of the string between the given point and the lowest point, to the tension at the lowest point.

(5) A uniform string hangs in equilibrium between two points. Prove that the square of the tension at any point is equal to the sum of the squares of the weight of the portion of the string between the given point and the lowest point, and of the tension at the lowest point.

(6) Show that, if a finite string of uniform density and thickness hang freely over two smooth pegs, the extremities of the string will be in the same horizontal line when the string is so placed as to be in equilibrium.

(7) A telegraph wire, weighing 400 lbs. per mile, is stretched between two points in the same horizontal line at a distance of 100

yds., with a horizontal tension of 400 pounds-weight. Find how much the lowest point of the wire will be below the fixed points.

Ans. 2·1... ft.

(8) Two light rings slide on a rough horizontal rod (angle of repose $=\alpha$). The ends of a heavy chain (length $=2l$) are attached to the rings. Obtain an equation to determine the greatest distance (d) at which the rings can rest apart.

Ans. $2=\tan\alpha\left(e^{\frac{d}{2l\tan\alpha}}-e^{-\frac{d}{2l\tan\alpha}}\right)$.

(9) A uniform wire weighing w lbs. per foot and just able to stand a stress of P pounds-weight is to be hung between two points in the same horizontal line, distant d ft., so as to be on the point of breaking. Obtain an equation to determine the length (l) of the wire.

Ans. $l=\dfrac{\sqrt{4P^2-w^2l^2}}{2w}\left(e^{\frac{dw}{\sqrt{4P^2-w^2l^2}}}-e^{-\frac{dw}{\sqrt{4P^2-w^2l^2}}}\right)$.

(10) A cord, 202 ft. long, 10 ft. of which weigh 1 lb., is hung between two points in the same horizontal line distant 200 ft. Obtain an equation to determine the tension (t) at the lowest point in terms of the weight of a pound.

Ans. $202=10t\left(e^{\frac{10}{t}}-e^{-\frac{10}{t}}\right)$. Solving this equation by a series of approximations, we find t to be about 40 lbs.-weight.

CHAPTER V.

DYNAMICS OF EXTENDED BODIES.

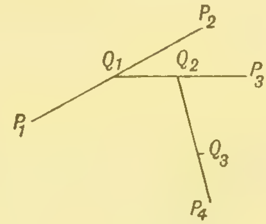
398. We shall consider next systems of particles which are so complex that it is impossible to determine the motions of the particles singly, as we did in the case of the simple systems of Chapter III.

An extended body, whether it be in the solid, liquid, or gaseous form, or consist of bodies in different forms, may be regarded as an assemblage of an indefinitely great number of particles.

The internal forces of such a system are those which act between the particles of the system themselves; the external forces are those which are exerted upon particles of the system by bodies which are not parts of the system. Thus, if we are considering the Solar System, the attractions of the sun on the planets and of the planets on one another are internal forces; the attractions of other heavenly bodies on the sun or planets are external forces.

399. *Centre of Mass.*—In studying systems of particles, we shall find it convenient to determine at the outset the position and properties of an important point called the Centre of Mass or Centre of Inertia.

If two particles of masses m_1 and m_2 occupy the positions P_1, P_2, and if the line P_1P_2 be divided at Q_1, so that



$$m_1 : m_2 = Q_1P_2 : Q_1P_1,$$

the point Q_1 is called the centre of mass of the two particles. If there be at P_3 a third particle, of mass m_3, and if Q_1P_3 be divided at Q_2, so that

$$m_1 + m_2 : m_3 = Q_2P_3 : Q_2Q_1,$$

the point Q_2 is called the centre of mass of the three particles. If there be at P_4 a fourth particle of mass m_4, and if Q_2P_4 be divided at Q_3, so that

$$m_1 + m_2 + m_3 : m_4 = Q_3P_4 : Q_3Q_2,$$

the point Q_3 is called the centre of mass of the four particles. If there be any number of particles in a given system, and if the above process be extended to all the particles of the system, the point thus determined is called the centre of mass of the system.

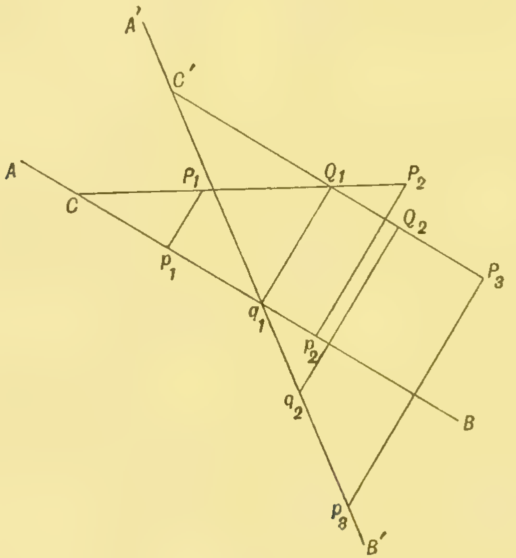
400. To determine the distance from a given plane of the centre of mass of a system of particles in terms of the masses of the particles and their distances from the same plane.—Let m_1, m_2, etc., be the masses, and P_1, P_2, etc., the positions, of the particles of the system, and let d_1, d_2, etc., be their distances from the given plane. P_1, P_2, etc., will not in general be in the same plane. Let a plane through P_1 and P_2, and perpendicular to the given plane, intersect it in AB; and let P_1P_2, produced if necessary, meet the given plane, and therefore AB, in C. The distances of P_1, P_2, from the given plane will be the perpendiculars P_1p_1, P_2p_2, from P_1 and P_2, on AB. From Q_1, the centre of mass of m_1 and m_2, draw a perpendicular to the given plane, intersecting AB therefore in a point q_1, and call Q_1q_1, δ_1.

We have by construction

$$
\begin{aligned}
m_1 : m_2 &= Q_1P_2 : Q_1P_1 \\
&= CP_2 - CQ_1 : CQ_1 - CP_1 \\
&= (CP_2 - CQ_1)\sin P_2CB : (CQ_1 - CP_1)\sin P_2CB \\
&= d_2 - \delta_1 : \delta_1 - d_1.
\end{aligned}
$$

Hence $$\delta_1 = \frac{m_1d_1 + m_2d_2}{m_1 + m_2}.$$

Let a plane through Q_1 and P_3 and perpendicular to the given plane intersect it in $A'B'$. If P_3 is in the same plane as P_1P_2 and AB, $A'B'$ will coincide with AB; if



not it will intersect it in q_1. From P_3 and Q_2 the centre of mass of m_1, m_2, and m_3, draw P_3p_3 and Q_2q_2 perpendiculars to the given plane, and therefore to $A'B'$, and let P_3Q_1 meet $A'B'$ in C'. Then as before, calling Q_2q_2, δ_2, we have

$$
\begin{aligned}
m_1 + m_2 : m_3 &= Q_2P_3 : Q_2Q_1 \\
&= (C'P_3 - C'Q_2)\sin P_3C'B' : (C'Q_2 - C'Q_1)\sin P_3C'B \\
&= d_3 - \delta_2 : \delta_2 - \delta_1.
\end{aligned}
$$

Hence $$\delta_2 = \frac{(m_1+m_2)\delta_1 + m_3 d_3}{m_1+m_2+m_3},$$

and, substituting for δ_1 its value as determined above,

$$\delta_2 = \frac{m_1 d_1 + m_2 d_2 + m_3 d_3}{m_1+m_2+m_3}.$$

Similarly, by extending the investigation to all the n particles of the system, we find, if Δ be the distance of the centre of mass of the system from the given plane,

$$\Delta = \frac{m_1 d_1 + m_2 d_2 + \text{etc.} + m_n d_n}{m_1 + m_2 + \text{etc.} + m_n} = \frac{\Sigma md}{\Sigma m};$$

i.e, the distance of the centre of mass of a system of particles from a given plane is equal to the sum of the products of the masses of the particles into their distances from the plane, divided by the sum of the masses of the particles.

401. As the order in which the particles are taken up in the above investigation affects only the order in which the various terms of the numerator and denominator of the above expression, $\Sigma md/\Sigma m$, are written, the centre of mass has the same distance from any given plane, *i.e.*, the same position, in whatever order the particles may be subjected to the process by which the point is determined.

402. The same result will be obtained if the particles of the system be divided into groups, and the centres of mass of the groups determined, and if the above process be then continued, the groups being imagined as replaced by particles situated at their centres of mass and having masses equal to the masses of the groups.

403. If the given plane pass through the centre of mass of the system, we will have $\Delta = 0$, and therefore $\Sigma md = 0$. Hence the sum of the products of the masses

of the particles of a system into their distances from a plane passing through the centre of mass is zero.

404. If x_1, y_1, z_1, x_2, y_2, z_2, etc., are the rectangular co-ordinates of particles of masses m_1, m_2, etc., and if $\bar{x}$, $\bar{y}$, $\bar{z}$ are the co-ordinates of their centre of mass, x_1, x_2, etc., $\bar{x}$ are (6) distances from the yz plane, y_1, y_2, etc., $\bar{y}$ distances from the xz plane, and z_1, z_2, etc., $\bar{z}$ distances from the xy plane. Hence

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},\quad \bar{y}=\frac{\Sigma my}{\Sigma m},\quad \bar{z}=\frac{\Sigma mz}{\Sigma m}.$$

These three equations determine the position of the centre of mass.

If the origin of co-ordinates coincide with the centre of mass, we have

$$\bar{x}=\bar{y}=\bar{z}=0,$$

and hence $$\Sigma mx=\Sigma my=\Sigma mz=0.$$

405. *Determination of Centres of Mass in Special Cases.*—In general the determination of the position of the centre of mass requires the use of the Integral Calculus to effect the necessary summation. In the case of some bodies, however, of simple geometrical form and uniform density, its position may be determined by elementary mathematical methods. Examples are given below (408).

406. *Centres of Mass of Homogeneous Symmetrical Bodies.*—If a homogeneous body is symmetrical about a point, a line, or a plane, its particles may be divided into pairs, the members of each of which are of equal mass and at equal distances from the point, line, or plane, respectively. The centres of mass of the various pairs are therefore in the point, line, or plane, and consequently also the centre of mass of the whole body. Hence the centre of mass of a uniform thin straight rod is its middle

point, that of a uniform thin circular rod its centre, that of a uniform thin rod bent in the form of a parallelogram the point of intersection of its diagonals, that of a uniform thin circular plate its centre, that of a uniform thin plate in the form of a parallelogram the point of intersection of its diagonals, that of a uniform spherical shell its centre, that of a parallelopiped the point of intersection of its diagonals, that of a circular cylinder with parallel ends the middle point of its axis, that of a sphere its centre, and so on.

407. *Centre of Mass of a Body, the Masses and Centres of Mass of whose Parts are known.*—Let m_1, m_2, etc., be the masses of the various portions of the body, x_1, y_1, z_1, x_2, y_2, z_2, etc., the co-ordinates of their centres of mass, and $\bar{x}$, $\bar{y}$, $\bar{z}$, the co-ordinates of the centre of mass of the body, then (402) we have

$$\bar{x}=\frac{m_1x_1+m_2x_2+\text{etc.}}{m_1+m_2+\text{etc.}},$$

and similar expressions for $\bar{y}$ and $\bar{z}$. By the aid of these expressions, the centre of mass of a part of a body also may be determined when the masses of all the parts are known, together with the centres of mass of the whole body and of all the parts except this one.

408. *Examples.*

(1) Four particles of 12, 11, 7, and 5 kilogrammes are placed in a line, their distances being 48, 48, and 42 cm. respectively. Find their centre of mass.

Ans. Distance from body of greatest mass = 54 cm.

(2) A line AB is bisected in C_1, C_1B in C_2, C_2B in C_3, and so on *ad infinitum*. Particles are placed at C_1, C_2, C_3, etc., of masses m, $m/2$, $m/2^2$, etc. Show that the distance from B of the centre of mass of the whole system is equal to one-third of AB.

(3) At A, B, C are three particles of equal mass. Show that, if AB be bisected in D and DC divided at E so that $DE=DC/3$, the point E is the centre of mass of the system.

(4) Show that, if O be the centre of mass of three particles of unequal masses P, Q, R situated at A, B, C,

$$\text{area } OBC : \text{area } OCA : \text{area } OAB = P : Q : R.$$

(5) Find the centre of mass of five equal particles at the angular points A, B, C, D, E of a regular hexagon $ABCDEF$.

Ans. On the line joining the centre O of the circumscribing circle with C, and at a distance from O equal to $OC/5$.

(6) At the corners A, B, C, D, E, F, G, H of a cube of 1 ft. edge, particles are placed of 1, 2, 3, 4, 5, 6, 7, 8 lbs. respectively. Find the centre of mass.

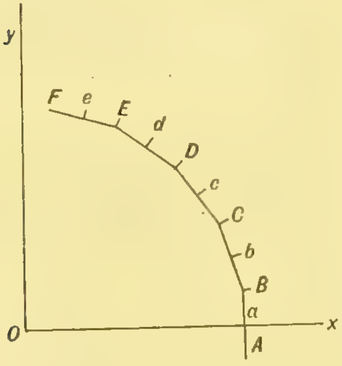
Ans. Distance from face $ABCD$, $\frac{13}{18}$ ft.; from $ABGF$, $\frac{5}{9}$ ft.; from $ADEF$, $\frac{5}{9}$ ft.

(7) A piece of uniform wire is bent twice at right angles so as to form three sides of a square of side a. Show that the distance of the centre of mass from the centre is $a/6$.

(8) Find the centre of mass of a uniform wire, bent into the form of a scalene triangle.

Ans. It is at the centre of the circle inscribed in the triangle formed by joining the middle points of the sides of the scalene triangle.

(9) Find the centre of mass of a uniform wire, bent so as to have the shape of n of the sides of a regular polygon.



Let $ABCDEF$ be the wire. Then the centres of mass of the portions AB, BC, etc., are at their middle points a, b, c, d, e. Let O be the centre of the inscribed circle and let its radius be r. Let ae subtend at O an angle 2α, and let ab, bc, etc., subtend each the angle θ. Then $2\alpha = (n-1)\theta$. Take Oa as axis of x and a line perpendicular to it as axis of y. Then the distances of a, b, c, etc., from Oy are r, $r\cos\theta$, $r\cos 2\theta$, etc., and their distances from Ox are 0, $r\sin\theta$, $r\sin 2\theta$, etc., respectively. Hence,

if $\bar{x}$, $\bar{y}$ are the distances from Oy, Ox respectively of the centre of mass of the wire,

$$\bar{x}=\frac{r}{n}[1+\cos\theta+\cos 2\theta+\text{etc.}+\cos(n-1)\theta]$$

$$=\frac{r}{n}\cdot\frac{\cos(\overline{n-1}/2)\theta\sin(n/2)\theta}{\sin(\theta/2)}*$$

$$=\frac{r}{n}\cdot\frac{\cos a\sin(n/\overline{n-1})a}{\sin(a/\overline{n-1})}.$$

$$\bar{y}=\frac{r}{n}[\sin\theta+\sin 2\theta+\text{etc.}+\sin(n-1)\theta]$$

$$=\frac{r}{n}\cdot\frac{\sin(\overline{n-1}/2)\theta\sin(n/2)\theta}{\sin(\theta/2)}*$$

$$=\frac{r}{n}\cdot\frac{\sin a\sin(n/\overline{n-1})a}{\sin(a/\overline{n-1})}.$$

Hence $\bar{y}/\bar{x}=\tan a$,

and $\sqrt{\bar{x}^2+\bar{y}^2}=\frac{r}{n}\cdot\frac{\sin(n/\overline{n-1})a}{\sin(a/\overline{n-1})}$,

i.e., the centre of mass is on a line through O whose inclination to the x axis is a, and is at a known distance from O.

(10) Find the centre of mass of a uniform wire in the form of a circular arc. [If n of Ex. 9 be made indefinitely great, and the sides of the polygon indefinitely short, $ABCDEF$ becomes a circular arc subtending at its centre an angle $2a$.]

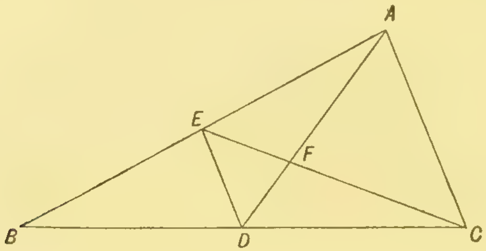
Ans. Distance from centre $=r\sin a/a$.

(11) The distance of the centre of mass of a uniform semi-circular wire of radius r from its centre is $2r/\pi$.

(12) Find the centre of mass of a uniformly thin homogeneous triangular plate.—Let ABC be such a triangular plate, and let it be divided by lines parallel to BC into an indefinitely great number of indefinitely narrow strips. Then the centre of mass of each strip is its middle point. Now the middle points of all these strips lie on the line AD drawn from A to D the middle point of BC. Hence

* See chapter on "Summation of Series" in any Plane Trigonometry.

the centre of mass of the plate is in the line AD. Similarly if E is the point of bisection of AB, the centre of mass lies in EC. Hence it is the point F in which AD and EC intersect.



Since E and D are the middle points of AB and BC, ED is parallel to AC. Hence the triangles AFC and DFE are similar, and

$$DF : FA = DE : AC = 1 : 2.$$

Hence the centre of mass is on the line DA, and at a distance from D equal to $DA/3$.

(13) Show that the centre of mass of the triangle* formed by joining the middle points of the sides of a triangle has the same position as that of the latter triangle.

(14) ABC is a triangle and D a fixed point in BC. A triangle BPC is cut away, whose vertex P is in AD. Show that whatever be the position of P, the centre of mass of the remainder lies on a fixed straight line.

(15) Given the base and perimeter of a triangle, show that the locus of its centre of mass is an ellipse.

(16) Prove that the centre of mass of the trapezoid formed by joining the middle points of two sides of a triangle is on the line joining their point of intersection to the middle point of the third side, at a point which is 2/9 of this line's length from the middle point of the third side.

(17) P is the point of intersection of the diagonals of a quadrilateral, Q the point which bisects the line joining the middle points of

* By the centre of mass of a surface is meant that of a uniformly thin homogeneous plate having the form of the surface.

these diagonals, and R a point in PQ produced, such that $QR = PQ/3$. Prove that R is the centre of mass of the quadrilateral.

(18) ABC is an isosceles right-angled triangle, right-angled at B. Squares are described on its three sides. Show that the distance of the centre of mass of a uniform thin plate of this form is at a distance from B equal to $13\sqrt{2}AB/27$.

(19) One circle touches another internally. The diameter of the latter is d, that of the former $\frac{2}{3}d$. Find the distance from the point of contact, of the centre of mass of the crescent or lune thus formed.

Ans. $\frac{19}{30}d$.

(20) Find the centre of mass of a sector of a circle of angle 2α and radius r.

If the curved portion of the boundary of the sector be divided into an indefinitely large number of equal arcs, the sector may be regarded as consisting of an indefinitely large number of equal isosceles triangles whose bases are the elements of the circular arc and whose equal sides are radii. The centre of mass of each of these triangles is at a distance $\frac{2}{3}r$ from the centre of the circle. Hence the centre of mass of the sector is the same as that of a circular arc of the angle 2α and the radius $\frac{2}{3}r$, and is therefore (Ex. 10) at a distance from the centre equal to $\frac{2}{3}r \sin \alpha/\alpha$.

(21) The centre of mass of a uniform thin semi-circular plate of radius r is at a distance from the centre equal to $4r/3\pi$.

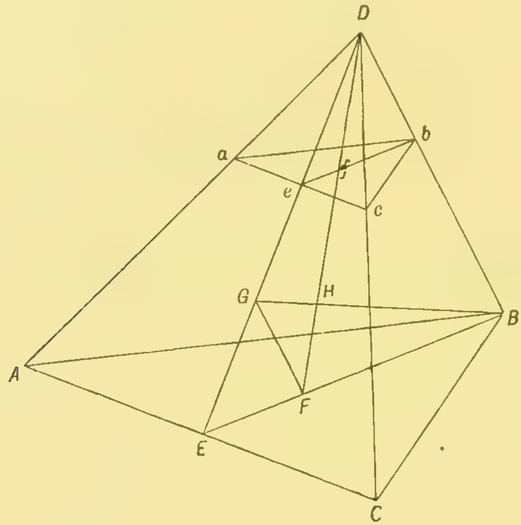
(22) The centre of mass of a uniform thin conical shell is on the axis, and at a distance from the vertex equal to $\frac{2}{3}$ of the height of the cone.

(23) Find the centre of mass of a homogeneous triangular pyramid.

Let the triangular pyramid $ABCD$ be divided by planes parallel to ABC into an indefinitely great number of indefinitely thin triangular plates of which abc is any one. Let F be the centre of mass of the plate ABC, and let the plane BDF intersect ABC, abc and ADC in BE, be, and DE respectively, and let DF and be, which are in the plane BDF, intersect in f. Since F is the centre

of mass of ABC, E is the middle point of AC. Since ac is parallel to AC, these lines being the intersections of parallel planes abc, ABC with the plane ADC,

$$ae : AE = De : DE = ec : EC.$$



Hence e is the middle point of ac. Since eb is parallel to EB, these lines being intersections of the parallel planes abc, ABC, with the plane BDE,

$$ef : EF = Df : DF = fb : FB.$$

Hence $ef = \frac{1}{3}eb$, and f is therefore the centre of mass of abc. Hence the centres of mass of all the triangular plates into which the pyramid is divided, and therefore the centre of mass of the pyramid, lie on the line DF. Similarly, if EG be equal to $\frac{1}{3}ED$, G will be the centre of mass of ACD, and the centre of mass of the pyramid will lie on the line GB. Now GB and DF are in the plane DBE, and intersect in H. Hence H is the centre of mass of the pyramid.

Since $EG : GD = EF : FB$, GF is parallel to DB, and the triangle GHF similar to the triangle BHD. Hence

$$FH : HD = FG : BD = EG : ED = 1 : 3.$$

Hence the centre of mass of the pyramid is on the line drawn from the centre of mass of the base to the vertex, and at a distance from the base equal to $\frac{1}{4}$ of the height of the pyramid.

(24) The centre of mass of a homogeneous pyramid with polygonal base, or of a cone with plane base, is on the line joining the centre of mass of the base to the apex, and at a distance from the base equal to $\frac{1}{4}$ of the height.

(25) The centre of mass of a homogeneous triangular pyramid coincides with that of four homogeneous spheres of equal mass, whose centres are at the four angles of the pyramid.

(26) The centre of mass of a homogeneous wedge, bounded by two similar, equal, and parallel triangular faces and by three rectangular faces, coincides with that of six equal particles placed at its angular points.

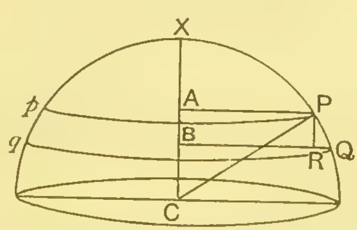
(27) A cube (edge $=1$) is truncated at one angle by a plane which bisects three adjacent edges. Show that the distance of the centre of mass of the remainder from the angle opposite to that which is cut off is $\frac{185}{376}\sqrt{3}$.

(28) From a right cone standing on a circular base another right cone is cut, standing on the same base, and the centre of mass of the remainder is at the vertex of the smaller cone. Show that this smaller cone is $\frac{1}{3}$ of the whole body.

(29) A cylindrical vessel of radius a stands vertically and contains water to a height h. A heavy sphere of radius $a/2$ is dropped into the water, and lies at the bottom of the vessel. Prove that the new centre of mass of the water lies somewhere within a circle whose radius is $a^2/12h$, and whose centre is at a distance $a/6 - 5a^2/72h$ from the old centre of mass of the water.

(30) Find the centre of mass of a zone of a uniform thin spherical shell.

A finite zone cut from a spherical surface (radius $= r$) by parallel planes (called bounding planes) may be divided into an infinitely large number of zones with bounding planes parallel to those of the finite zone but at infinitesimal distances each from the next. Let $PQ\,qp$ be one such zone. Its area may be put equal to $2\pi\, AP\,.\,PQ$, PQ being the intersection (ultimately rectilinear) of the zone by a



great circle of the sphere through the axis CX of the zone, C the centre of the sphere and AP the perpendicular from P on the axis. Draw QB also perpendicular to the axis and PR perpendicular to QB and therefore obviously parallel to the axis. As PC and PR are perpendicular to PQ and PA respectively, the right-angled triangles PRQ and PAC are similar. Hence $AP:PC=PR:PQ=AB:PQ$, and $AP.PQ=PC.AB$. Hence the area of the zone is $2\pi r.AB$. Its centre of mass is obviously on the axis, and may be considered as being at A or B or any intermediate point, A and B being ultimately coincident.

If the distances between the bounding planes of the successive infinitesimal constituents of the finite zone are equal, their areas and therefore their masses will be equal, and their centres of mass will therefore be situated on the axis at equal distances each from the next, extending from the one to the other of the two bounding planes. Hence the centre of mass of the zone will be on the axis and midway between its bounding planes.

(31) Find the centre of mass of a sector of a uniform sphere, and of a uniform hemisphere.

The sector may be imagined as consisting of an infinitely large number of equal infinitesimal pyramids. Their centres of mass are all (Ex. 23) at a distance from their bases equal to $\frac{1}{4}$ of their height, and therefore at a distance from the centre of the sphere equal to $\frac{3}{4}$ of the radius r. The centre of mass of the sector is therefore coincident with that of the zone of a spherical shell of radius $3r/4$, marked off by the radial bounding lines of the sector. The centre of mass of this zone is on its axis (the axis of the sector) and midway between its bounding planes (Ex. 30), and therefore at a distance from the centre of the sphere equal to

$$\frac{1}{2}\left(\frac{3r}{4}+\frac{3r}{4}\cos\theta\right)=\frac{3r}{8}(1+\cos\theta),$$

θ being half the semi-vertical angle of the sector.

In the case of the uniform hemisphere, $\theta=\pi/2$. Hence the distance of its centre of mass from the centre is $3r/8$.

409. *Velocity of the Centre of Mass.*—The component velocity, in a given direction, of the centre of mass of a

system of particles is equal to the sum of the products of the masses of the particles into their component velocities in the same direction, divided by the sum of the masses of the particles.

Let m_1, m_2, etc., be the masses of the particles of the system, s_1, s_2, etc., $\bar{s}$ the distances from a plane perpendicular to the given direction, of the particles and their centre of mass respectively, at a given instant, s_1', s_2', etc., $\bar{s}'$, their respective distances after a short time t. Then

$$\bar{s}=\frac{m_1s_1+m_2s_2+\text{etc.}}{m_1+m_2+\text{etc.}}; \quad \bar{s}'=\frac{m_1s_1'+m_2s_2'+\text{etc.}}{m_1+m_2+\text{etc.}}.$$

Hence, subtracting, and dividing by t,

$$(\bar{s}'-\bar{s})/t=\frac{m_1(s_1'-s_1)/t+m_2(s_2'-s_2)/t+\text{etc.}}{m_1+m_2+\text{etc.}},$$

or (44 and 101), $\bar{v}$ being the velocity of the centre of mass in the given direction,

$$\bar{v}=\dot{\bar{s}}=\frac{\Sigma m\dot{s}}{\Sigma m}.$$

This result may be otherwise expressed thus:—The velocity of the centre of mass in a given direction is equal to the momentum of the system (*i.e.*, to the algebraic sum of the momenta of the various particles) in the given direction, divided by the mass of the system.

410. It follows that the momentum of the system in any given direction relative to the centre of mass is zero. For from 409 we have

$$\Sigma m\dot{s}-\dot{\bar{s}}\Sigma m=\Sigma m(\dot{s}-\dot{\bar{s}})=0;$$

and from 96 (3) it is obvious that $\Sigma m(\dot{s}-\dot{\bar{s}})$ is the momentum in the given direction relative to the centre of mass.

411. *Acceleration of Centre of Mass.*—The component acceleration, in a given direction, of the centre of mass of

a system of particles is equal to the sum of the products of the masses of the particles into their component accelerations in the same direction, divided by the sum of the masses of the particles.

The proof may be left to the reader. It is similar to that of 409, $\bar{s}$, s_1, etc., being replaced by $\ddot{\bar{s}}$, $\ddot{s}_1$, etc. Hence (118), $\bar{a}$ being the acceleration of the centre of mass in the given direction,

$$\bar{a} = \ddot{\bar{s}} = \frac{\Sigma m\ddot{s}}{\Sigma m}.$$

412. It follows, as in 410, that the sum of the products of the masses of the particles of a system into their component accelerations, in any given direction, relative to the centre of mass, is zero.

413. *Examples.*

(1) If two particles move with uniform speed in straight lines, their centre of mass will either be at rest or will move with uniform speed in a straight line also.

(2) A number of particles of masses, m_1, m_2, etc., are projected at the same instant vertically upwards from given positions with given speeds, v_1, v_2, etc., respectively. Find (a) how long, and (b) how far, their centre of mass will rise.

Ans. (a) $\frac{1}{g}\frac{\Sigma mv}{\Sigma m}$; ($b$) $\frac{1}{2g}\left(\frac{\Sigma mv}{\Sigma m}\right)^2$.

(3) Two particles connected by a string are placed on two smooth inclined planes, the string passing over a smooth peg at the common summit of the planes. Show that the path of their centre of mass is the straight line which joins them when they are in such a position that the parts of the string on the two planes are to one another as the masses of the particles at their extremities, and that that particle will descend which in this position is the lower of the two.

(4) Of three equal particles which start from the highest point of a vertical circle, one drops down the vertical diameter, and the others slide down chords of 60° and 120° respectively, on the same

side of the diameter. Show that the centre of mass slides down a chord of $\cos^{-1}(-\frac{1}{1}\frac{3}{9})$, and that its rate of change of speed is $\frac{g}{6}\sqrt{19}$.

414. *Acceleration of Centre of Mass in terms of External Forces.*—The component acceleration, in any given direction, of the centre of mass of a system of particles is the same as the acceleration of a particle of mass equal to the mass of the system, acted on by a force in the given direction equal to the sum of the components in that direction of the external forces acting on the particles of the system.

Let F_1, F_2, etc., be the components, in the given direction, of the resultants of all the external forces acting on the particles (masses $= m_1, m_2$, etc.); and let F_1', F_2', etc., be the components, in the same direction, of the resultants of all the internal forces acting on m_1, m_2, etc., respectively. Then s_1, s_2, etc., being the distances from a plane perpendicular to the given direction, we have (317 and 318)

$$F_1 + F_1' = m_1\ddot{s}_1; \quad F_2 + F_2' = m_2\ddot{s}_2; \quad \text{etc.}$$

Hence
$$\Sigma F + \Sigma F' = \Sigma m\ddot{s}.$$

Now by the third law of motion, the internal forces consist of pairs of equal and opposite forces, whose sum is therefore zero. Hence $\Sigma F' = 0$, and $\Sigma F = \Sigma m\ddot{s}$. Now (411) $\ddot{\bar{s}} = \Sigma m\ddot{s}/\Sigma m$. Hence, calling $\bar{a}$ the acceleration, in the given direction, of the centre of mass,

$$\bar{a} = \ddot{\bar{s}} = \frac{\Sigma F}{\Sigma m}.$$

And it follows from 317 that $\bar{a}$, as determined by this formula, is the acceleration that a particle of mass Σm would have, if acted on by a force equal to ΣF.

If therefore the external forces acting on the system and the mass of the system are known, the acceleration of the centre of mass may be determined.

415. If the components of the external forces in three rectangular directions, the axes of x, y, z, are X_1, X_2, etc., Y_1, Y_2, etc., Z_1, Z_2, etc., respectively, the component accelerations of the centre of mass in these directions are

$$\ddot{\bar{x}}=\frac{\Sigma X}{\Sigma m},\quad \ddot{\bar{y}}=\frac{\Sigma Y}{\Sigma m},\quad \ddot{\bar{z}}=\frac{\Sigma Z}{\Sigma m};$$

and the component accelerations being known, the magnitude and direction of the resultant acceleration may be determined.

416. In the special case in which a system of particles is acted upon by external forces, the sum of whose components in any given direction is zero, the acceleration of the centre of mass is zero. For ΣF being zero, so also is $\ddot{\bar{s}}$.

It follows also from 411 that if $\ddot{\bar{s}}$ is zero, $\Sigma m\ddot{s}$ is zero also, and $\Sigma m\ddot{s}$ may easily be shown to be the rate of change with time of the momentum of the system, in the given direction. Hence if there are no external forces the momentum of the system is constant. This result is often spoken of as the *Principle of the Conservation of Linear Momentum.*

417. [*D'Alembert's Principle.*—In 1742 D'Alembert proposed as a law of motion what is called his Principle. It is usually enunciated in the following form, though this is not the form in which it was enunciated by D'Alembert himself:—

The impressed forces, with the reversed effective forces, of a system of material particles, constitute together a system of forces in equilibrium.

By the term "impressed force" is meant an external force acting on the system. The "effective force" on a particle was the name given to the product of its mass into its acceleration, and this hypothetical force was supposed to act in the direction of the acceleration. A

reversed effective force would thus act in the opposite direction. If F is the component in a given direction of the resultant of all the external forces acting on a particle, m its mass and a the component in the given direction of its acceleration, ΣF is the sum of all such components of external forces, Σma the sum of the components, in the given direction, of the effective forces, and $-\Sigma ma$ therefore the sum of the components in the given direction of all the reversed effective forces. Since the sum of the components in a given direction of all the forces of a system of forces in equilibrium (323 and 326) is zero, D'Alembert's principle may be expressed by the equation $\Sigma F - \Sigma ma = 0$. This equation is obviously that obtained in 414; and thus D'Alembert's principle may be deduced immediately from Newton's second and third laws, which were formulated in 1687.

By D'Alembert's principle every kinetic problem was reduced to one of equilibrium between actual and fictitious forces. It was thus of great practical importance, as enabling the equations of motion to be written down for any system for which the conditions of equilibrium had been investigated.]

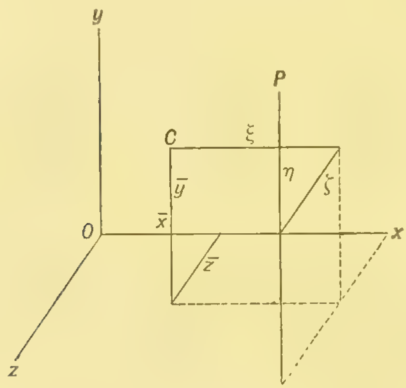
418. *The Moment of Momentum* of a particle about a given axis is the product of its mass into the moment of its velocity (104) about the axis. If v be the component velocity of the particle of mass m, in a plane perpendicular to the given axis, and p the distance from the axis of a line drawn through the position of the particle, representing v, the moment of momentum of the particle is mvp, the sign being determined according to the convention of 103.

The algebraic sum of the moments of momentum of all the particles of a system about a given axis, is the moment of momentum of the system about that axis.

If the given axis be taken as axis of z, the analytical expression for the moment of momentum of the particle

will be (106), $m(\dot{y}x-\dot{x}y)$, and for that of a system of particles, $\Sigma m(\dot{y}x-\dot{x}y)$.

419. The moment of momentum of a system of particles about a given axis is equal to its value about a parallel axis through the centre of mass, together with the moment of momentum, about the given axis, of a particle of mass equal to the mass of the system, and situated at, and having the acceleration of, the centre of mass.



Take the given axis as axis of z, and Ox, Oy, as the other rectangular axes of co-ordinates. Let x, y, z be the co-ordinates of any particle of mass m, at P, and $\bar{x}$, $\bar{y}$, $\bar{z}$, those of the centre of mass C. Let P's distance from C in the directions of Ox, Oy, Oz, be ξ, η, ζ. Then

$$x=\bar{x}+\xi,\quad y=\bar{y}+\eta.$$

Also (96)

$$\dot{x}=\dot{\bar{x}}+\dot{\xi},\quad \dot{y}=\dot{\bar{y}}+\dot{\eta}.$$

Hence the moment of momentum of the system

$$\begin{aligned}\Sigma m(\dot{y}x-\dot{x}y)&=\Sigma m\{(\dot{\bar{y}}+\dot{\eta})(\bar{x}+\xi)-(\dot{\bar{x}}+\dot{\xi})(\bar{y}+\eta)\}\\ &=(\dot{\bar{y}}\bar{x}-\dot{\bar{x}}\bar{y})\Sigma m+\Sigma m(\dot{\eta}\xi-\dot{\xi}\eta)\\ &\qquad+\bar{x}\Sigma m\dot{\eta}+\dot{\bar{y}}\Sigma m\xi-\dot{\bar{x}}\Sigma m\eta-\bar{y}\Sigma m\dot{\xi}\\ &=(\dot{\bar{y}}\bar{x}-\dot{\bar{x}}\bar{y})\Sigma m+\Sigma m(\dot{\eta}\xi-\dot{\xi}\eta),\end{aligned}$$

since (404 and 410), each of the quantities $\Sigma m\eta$, $\Sigma m\xi$, $\Sigma m\dot{\eta}$, and $\Sigma m\dot{\xi}$ is equal to zero. $\Sigma m(\dot{\eta}\xi-\dot{\xi}\eta)$ is clearly the moment of momentum of the system about an axis, parallel to the given axis, through the centre of mass, and $(\dot{\bar{y}}\bar{x}-\dot{\bar{x}}\bar{y})\Sigma m$ is equal to the moment of momentum of a particle of mass Σm, whose co-ordinates and component velocities are those of the centre of mass.

420. *The Angular Momentum* of a particle about a given axis is the product of its mass (m) into its angular velocity (ω) about the given axis, into the square of its distance (r) from the given axis—in symbols $m\omega r^2$.

The algebraic sum of the angular momenta of all the particles of a system about a given axis is the angular momentum of the system about that axis.

421. It follows from 132 that $mvp = m\omega r^2$. Hence the moment of momentum about any given axis is equal to the angular momentum about the same axis of either a single particle or a system of particles. Hence also the rate of change of angular momentum about any given axis is equal to the rate of change of moment of momentum about the same axis.

422. *The Moment of the Acceleration of Momentum* of a particle about a given axis is the product of its mass into the moment of its acceleration about the given axis. The algebraic sum of all such products for all the particles of a system is the moment of the acceleration of momentum for the system. If a be the component acceleration of a particle in a plane perpendicular to the given axis, and if p be the distance from the axis of a line drawn through the position of the particle representing a, the moment of the acceleration of momentum for the particle is map and for the system Σmap. The analytical expression for it will be (123 and 106) $\Sigma m(\ddot{y}x - \ddot{x}y)$.

423. It follows from 124 that the moment of the acceleration of momentum of a particle about a given axis is equal to the product of its mass into the rate of change of the moment of its velocity, and therefore to the rate of change of its moment of momentum, and therefore to the rate of change of its angular momentum about the same axis. In symbols $map = m(\dot{\overline{\omega r^2}})$. Hence also for a system of particles $\Sigma map = \Sigma m(\dot{\overline{\omega r^2}})$.

424. It may be shown by the method employed in 419 that the moment of the acceleration of momentum of a system about a given axis is equal to its value about a parallel axis through the centre of mass, together with the moment of the acceleration of momentum about the given axis, of a particle, having a mass equal to the mass of the system, and situated at, and having the acceleration of, the centre of mass. With the symbols of 419,

$$\Sigma m(\ddot{y}x-\ddot{x}y)=(\ddot{\bar{y}}\bar{x}-\ddot{\bar{x}}\bar{y})\Sigma m+\Sigma m(\ddot{\eta}\xi-\ddot{\xi}\eta).$$

425. *The Moment of a Force* about a line or axis is the product of the component of the force in a plane perpendicular to the axis, into the distance from the axis of the line of action of the component. If F is the magnitude of the component, and p its distance from the given axis, Fp is the magnitude of the moment of the force about the axis. Its sign is determined by a convention similar to that of 103.

426. It follows from 313 and 107 that the moment of a force is equal to the algebraic sum of the moments of its components about any fixed axis.

427. If the given line be taken as axis of z, and other lines perpendicular to it and to one another (as in 419) as axes of x and y, and if x, y, and z be the co-ordinates of the particle on which the force acts, and X, Y, Z the components, in the directions of the axes, of the given force, then X and Y are rectangular components of F in the plane perpendicular to the given line, and y and x their respective distances from the given line, and therefore Yx and $-Xy$ their respective moments about it. Hence (426) $Fp=Yx-Xy$. This is the analytical expression for the moment of a force.

428. The sum of the moments, about an axis fixed in space, of the external forces acting on a system of particles

is equal to the rate of change of the angular momentum of the system about the given axis.

R being the component, in a plane perpendicular to the given axis, of the resultant force acting on the particle, a its component acceleration in the same plane, and m its mass, we have $R=ma$. If p is the common distance of R and a from the given axis, $Rp=map$. If F and F' are the components in the same plane of the resultants of the external and internal forces respectively, acting on m, F and F' are components of R. If therefore P and P' are their respective distances from the given axis (426),

$$Rp=FP+F'P'=map.$$

Hence, for the system (423),

$$\Sigma FP+\Sigma F'P'=\Sigma map=\Sigma m\,(\dot{\overline{\omega r^2}}).$$

Now the internal forces consist of pairs of equal and opposite forces equidistant from the axis. Hence

$$\Sigma F'P'=0,$$

and

$$\Sigma FP=\Sigma m\,(\dot{\overline{\omega r^2}}).$$

The analytical expression of this result is (427 and 422)

$$\Sigma(Yx-Xy)=\Sigma m(\ddot{y}x-\ddot{x}y).$$

429. In the special case in which the sum of the moments of the external forces about the given axis is zero, the angular momentum of the system about the given axis is constant. For we have $\Sigma FP=\Sigma m\,(\dot{\overline{\omega r^2}})=0$. Hence $\Sigma m\omega r^2$ is constant. This result is called the principle of the *conservation of angular momentum.*

It follows that $\Sigma m\omega r^2/2=\text{constant}$. Now (133) $\omega r^2/2$ is the area swept over per unit of time by the radius vector of the particle of mass m. Hence the above result is also called the principle of the *conservation of areas.*

430. From 428 and 424 it follows that

$$\Sigma(Yx - Xy) = (\ddot{\bar{y}}\bar{x} - \ddot{\bar{x}}\bar{y})\Sigma m + \Sigma m(\ddot{\eta}\xi - \ddot{\xi}\eta).$$

This applies, as we have seen, to an axis fixed in space. If we choose the axis so that at the instant under consideration it is passing through the centre of mass of the system, we have $\bar{x} = \bar{y} = 0$, $x = \xi$, and $y = \eta$, and therefore

$$\Sigma(Y\xi - X\eta) = \Sigma m(\ddot{\eta}\xi - \ddot{\xi}\eta).$$

Now $\Sigma m(\ddot{\eta}\xi - \ddot{\xi}\eta)$ is the rate of change of angular momentum about an axis parallel to the given axis through the centre of mass; and $\Sigma(Y\xi - X\eta)$ is (427 and 428) the value the rate of change of angular momentum would have if the centre of mass were fixed. Hence the rate of change of angular momentum about the centre of mass, produced in a system of particles by the forces to which it is subjected, is the same as that which would be produced if the centre of mass were fixed.

431. *Equations of Motion.*—We have now obtained two important results; the first, that of 414, by which the acceleration of the centre of mass of a system is expressed in terms of the external forces acting on it, and its mass; and the second, that of 428, by which the rate of change of angular momentum of a system about a fixed axis (or, 430, about an axis through the centre of mass) is expressed also in terms of the external forces and the mass. These equations tell us all that we can know of the motion of a system of particles without data as to the internal forces. They are therefore called the equations of motion of a system of particles or of an extended body. The principles of the conservation of linear and angular momentum are special cases of these equations of motion.

432. *Energy of a system of Particles.*—The energy of a system of particles is the sum of the amounts of work power which the particles possess individually and the amounts which they may possess collectively.

The kinetic energy depends only on the masses and velocities of the individual particles, and will be equal to $\Sigma \frac{1}{2} m V^2$.

The potential energy will depend upon the mutual forces and the distances between the particles, as well as upon the external forces acting on them and their position relative to the bodies by which such forces are exerted.

We have seen (347) that if a particle move in the neighbourhood of a centre of force, its potential energy will be increased by the amount of the work done against the force during the motion. If the centre of force is an attracting particle, the two particles must by the Third Law be acted upon at all stages of the motion by equal and opposite forces. If both particles move work will thus be done on both. To find the change of potential energy, we may imagine their motions to be divided up each into an indefinitely large number of infinitesimal portions, and we may think of the particles as describing alternately single infinitesimal portions of their respective paths, thus describing their paths simultaneously. During each infinitesimal motion, the potential energy of the moving particle will increase by an amount depending on the change of distance of the two and equal to the work done against the mutual attraction. Hence during the whole finite relative motion of the two the potential energy of the two together will increase by an amount depending on the change of distance between them and equal to the work done against the mutual attraction during the motion. The potential energy thus gained cannot be ascribed wholly to either particle, but must be ascribed to the system of the two particles. The potential energy of the system of two particles will thus increase during any relative motion by an amount depending only on the initial and final relative positions and equal to the work done against the mutual attraction during the motion by which the change of relative position was brought about.

We may apply the same mode of treatment to a system of any number of particles, imagining their motions to be divided up into infinitesimal portions and to be conducted step by step as above so that they may be considered as occurring simultaneously. During each small motion of each particle, the potential energy of the moving particle increases by an amount equal to the work done against the various forces exerted upon it by the other particles, and therefore, if we assume that the mere presence of other particles does not affect the mutual force between any two, dependent only on the change of distance of the moving particle from the other particles. Hence during the whole finite change of configuration of the system the potential energy of all the particles taken together will increase by an amount equal to the work done against the mutual forces and dependent only on the initial and final configurations. As in the case of the system of two particles, the potential energy gained must be ascribed to the system, not to individual particles.

If we choose some convenient configuration of the system as the configuration of zero potential energy, the potential energy of the system in any other given configuration will be equal to the work done against the internal forces of the system during the passage of the system from the configuration of zero potential energy to the given configuration.

If the particles of the system are acted upon by external forces, which also are central forces, its potential energy will be increased during any change of configuration by the amount of the work done against them also. The potential energy of the system will therefore depend upon the configuration of the larger system including the bodies which exert the external forces.

433. The work done by the forces acting on the particles of a system, during any change of configuration, is

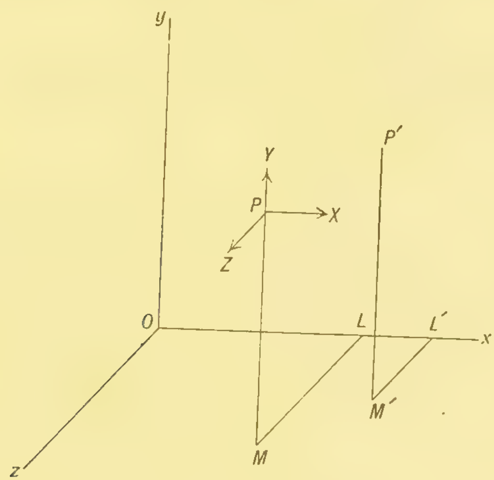
equal to the increment of the kinetic energy of the system.

Let X, Y, Z be the components, in three rectangular directions Ox, Oy, Oz, of the resultant of the external and internal forces acting on a particle of mass m at a point P, whose co-ordinates are OL, MP, LM, or x, y, z. Then (414) taking all the particles of the system into consideration, we have

$$\Sigma X = \Sigma m\ddot{x}, \quad \Sigma Y = \Sigma m\ddot{y}, \quad \Sigma Z = \Sigma m\ddot{z};$$

and therefore

$$\Sigma(X - m\ddot{x}) = 0, \quad \Sigma(Y - m\ddot{y}) = 0, \quad \Sigma(Z - m\ddot{z}) = 0.$$



Let the particle undergo any indefinitely small displacement, say to a point P', whose co-ordinates are OL', $M'P'$, $L'M'$, or x', y', z'. The displacement being indefinitely small, the resultant force acting on the particle may be considered uniform. The components of the displacement in the axes of x, y, and z will be $x'-x$, $y'-y$, $z'-z$. Multiplying by these components, we have

$$\Sigma(X - m\ddot{x})(x'-x) = 0, \quad \Sigma(Y - m\ddot{y})(y'-y) = 0,$$
$$\Sigma(Z - m\ddot{z})(z'-z) = 0.$$

Hence, also,

$$\Sigma(X-m\ddot{x})(x'-x)+\Sigma(Y-m\ddot{y})(y'-y)$$
$$+\Sigma(Z-m\ddot{z})(z'-z)=0,$$

and

$$\Sigma\{X(x'-x)+Y(y'-y)+Z(z'-z)-m[\ddot{x}(x'-x)+\ddot{y}(y'-y)$$
$$+\ddot{z}(z'-z)]\}=0.$$

The terms, $\ddot{x}(x'-x)$, $\ddot{y}(y'-y)$, $\ddot{z}(z'-z)$, being products of component accelerations into indefinitely small component displacements, during which the accelerations may be considered to be constant, may be put equal to

$$\tfrac{1}{2}(\dot{x}'^2-\dot{x}^2),\ \ \tfrac{1}{2}(\dot{y}'^2-\dot{y}^2),\ \ \tfrac{1}{2}(\dot{z}'^2-\dot{z}^2).$$

Hence

$$\Sigma\left\{X(x'-x)+Y(y'-y)+Z(z'-z)-\frac{m}{2}[\dot{x}'^2+\dot{y}'^2+\dot{z}'^2\right.$$
$$\left.-(\dot{x}^2+\dot{y}^2+\dot{z}^2)]\right\}=0.$$

If W denote the work done on the particle m, v the velocity of the particle at P, and v' its velocity at P', we have (342, 98, and 88),

$$W=X(x'-x)+Y(y'-y)+Z(z'-z),$$
$$v'^2=\dot{x}'^2+\dot{y}'^2+\dot{z}'^2,$$

and $$v^2=\dot{x}^2+\dot{y}^2+\dot{z}^2.$$

Hence $$\Sigma\left\{W-\frac{m}{2}(v'^2-v^2)\right\}=0.$$

Hence the sum of all the quantities of work done by the forces acting on the particles of a system, during any indefinitely small change of configuration, is equal to the sum of the quantities of kinetic energy gained by the particles.

Any finite change of the configuration of a system may be broken up into an indefinitely large number of indefinitely small changes, to each of which an equation

similar to the above applies. Adding them, we have for a finite change of configuration,

$$\Sigma\Sigma\left\{W-\frac{m}{2}(v'^2-v^2)\right\}=0,$$

or the work done by the acting forces during any finite change of configuration of a system of particles is equal to the increment of the kinetic energy of the system.

434. *Conservation of Energy.*—If all the forces (external and internal) are central forces, the work done against them results (432) in the increment of the potential energy of the system. The amount of the potential energy produced is equal to the work thus done *against* the forces, and is therefore equal to *minus* the work done *by* them. If therefore P is the increment of potential energy of a particle in any small displacement, $P=-W$; and hence

$$\Sigma\Sigma\{-P-\tfrac{1}{2}m(v'^2-v^2)\}=0;$$

and if K denote the increment of kinetic energy,

$$\Sigma\Sigma(P+K)=0.$$

Hence the sum of the potential and kinetic energies of a system of particles is constant, provided all the acting forces are central forces dependent only on the positions of the particles. This result is called the law of the Conservation of Energy. A system of particles to which it applies is called a conservative system.

435. A cycle of transformations of a system is a series of changes of configuration by which the particles are brought finally to their initial positions. If the system is conservative and isolated, it is clear that the initial and final potential and kinetic energies must be the same. If therefore a conservative system of particles be so arranged that when set in motion it undergoes cyclical transformations, the cycles of transformations will go on for ever. If, for example, heavenly bodies moving in

space met with no resistance to their motion, of the nature of friction, the solar system would form a system of this kind and the planets must continue to move round the sun for ever. If we had materials of perfect smoothness and with other properties excluding the possibility of the action of non-conservative forces, it would be possible to make a machine which, once started, would run for ever without work being done upon it, provided work were not done by it.

An isolated conservative system thus undergoing cycles of transformation can never, however, increase the total quantity of its energy. If therefore natural forces are of the conservative kind, it will be impossible to devise a machine which, when set in motion and left to itself, will both run itself and do external work—in other words, the "perpetual motion" will be an impossibility.

436. *Law of Energy.*—If any of the forces acting on the particles of the system are of the nature of resistances which depend upon the velocity of a particle, not on its position merely, the work done against them does not result in the production of potential energy. In such systems therefore, which are at any rate apparently non-conservative, the work done against the acting forces is equal to the sum of the increment of potential energy and of the work done against such resistances. If this work be denoted by w, we have therefore $P+w=-W$, and hence

$$\Sigma\Sigma(P+w+K)=0.$$

Hence the kinetic and potential energy of a material system, together with the energy expended in overcoming friction and other forms of non-conservative force, is a constant quantity.

This result is the general law of energy, of which the law of the conservation of energy is a special case.

437. If therefore a non-conservative system of particles be so arranged that, when set in motion, it undergoes

cyclical transformations, its energy will gradually diminish, and its cyclical transformations cannot therefore go on for ever. It is possible that the planets move in a resisting medium, whose resistance they expend energy in overcoming. If so, they must be moving in spiral paths and getting gradually nearer the sun. No machine can be constructed whose parts in their relative motions do not meet with frictional resistance and other forms of (apparently) non-conservative force. Hence no machine can be constructed which will run itself even if no external work be done.

438. When work is done against forces, such as friction, which are, apparently at least, non-conservative, there seems at first sight to be no return in the form of energy. Experiment, however, has shown that when energy is thus expended heat is always produced, that heat is a form of energy, and that the amount of thermal energy produced when work is done against friction or other such forces, is the exact equivalent of the work so done. Hence the law of energy in the case of material systems which are apparently non-conservative may be thus expressed:

The energy of the system, including kinetic, potential, and thermal energy, is a constant quantity, if the system is isolated, so that it can neither give energy to, nor receive energy from, outside bodies.

439. The frictional and other non-conservative forces which we find acting on bodies in their relative motions, and to whose action is due the apparent non-conservative character of material systems, are observed to act between bodies of finite size. It is possible therefore that, if we could observe all the motions of the particles or small parts of bodies, their apparent non-conservative character might disappear. When work is done against friction, for example, it may be that the relative motions of the

particles of the bodies in contact are increased, so that though the rubbing bodies do not gain potential energy their particles gain kinetic energy. Thermal energy is generally believed, though not yet proved, to be the kinetic energy of the particles of a body due to their motion among one another. If so, the laws of Thermodynamics should be capable of deduction from the laws of motion. At present however we do not know enough about the relative motions of the particles of a body, or how they are affected when the body meets with frictional or other such resistances, to make this deduction.

440. We have seen (433) that it is possible by the aid of the 2nd and 3rd Laws of Motion to prove the Law of the Conservation of Energy to hold for systems acted upon by central forces only, the energy referred to being what is frequently called mechanical energy, viz., the kinetic and potential energy which we have come upon in our study of dynamics. We have seen also (439) that, as shown by experiment, when work is done against such forces as friction, of which it is not known that they may be regarded as central forces, the work so done results in the production of an equivalent amount of other forms of energy, such as heat, and that the law of conservation, if all forms of energy are taken into account, holds generally.

In order to make the law of conservation in all its generality a law of Abstract Dynamics, it is necessary to add to the hypotheses embodied in the Laws of Motion some such hypothesis as that all natural forces may be regarded as central forces. The four laws which would thus form the basis of our subject would not however be independent. The First we have seen (289) to be a particular case of the Second. The Third is clearly a partial statement of the Fourth; for the assumption that natural forces are in all cases central forces involves not merely that action and reaction are equal and opposite, but also

that they are in the line joining the particles and dependent as to magnitude only on the distance between them. Thus the four laws would reduce to two: (1) The Law of Force—Newton's Second Law of Motion; and (2) The Law of Stress—that natural forces may be considered to be attractions or repulsions between particles, with magnitudes varying only with the distance between the particles.

It will be noticed that underlying these laws there is the fundamental assumption that bodies may be regarded as consisting of particles which exert forces upon one another at a distance. We have been making this assumption in dealing with extended bodies. If it be not admitted, or if it be desired, as in the discussion of the change of form or volume of bodies, to regard them as continuous and thus consisting of parts in contact, which exert forces only on neighbouring parts, the Law of Stress would need to be put into a different form.

Some writers accept the universal failure of efforts to discover the "perpetual motion," *i.e.*, to devise a machine capable both of running itself and of doing external work without having energy supplied to it, as proving the conservation of energy in all its generality, and make either the impossibility of the perpetual motion or the conservation of energy itself an additional dynamical hypothesis. It will be obvious however from 433 that the Law of the Conservation of Energy involves the 2nd Law of Motion, and that the Laws of Motion and the Law of the Conservation of Energy are not independent hypotheses. From a practical point of view the matter is of no great consequence. But if we wish, in formulating the hypotheses of dynamics, both to include among them the conservation of energy and to indicate the number of independent assumptions made, it would be necessary to recast Newton's Laws, and, as pointed out by Tait, to assume in addition to the conservation of energy (which would include Newton's Second Law) a

law of the transformation of energy (which would embody the Third Law).

441. In applying the law of energy, obtained above, to the solution of problems on the motion of material systems, it is important to notice that forces acting on fixed portions of the system, stresses between particles whose distances are invariable, and forces acting on particles whose motion is normal to the direction of the force, do no work and therefore do not appear in the equation of energy.

442. In the solution of such problems the following proposition will be of use to facilitate the calculation of the kinetic energy of the system:

The kinetic energy of a system of particles is equal to the sum of the kinetic energies of the particles of the system moving with velocities equal to their velocities relative to the centre of mass, together with that of a particle having a mass equal to the mass of the system and a velocity equal to the velocity of the centre of mass.

Ox, Oy, Oz being rectangular axes, let the co-ordinates of a particle of mass m be x, y, z. Then its component velocities are $\dot{x}, \dot{y}, \dot{z}$ and its kinetic energy $\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$. The kinetic energy of the system is thus $\Sigma\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of mass and ξ, η, ζ the distances of the particle from it in the directions of Ox, Oy, Oz respectively. Then (419)

$$x=\bar{x}+\xi,\quad y=\bar{y}+\eta,\quad \text{and}\quad z=\bar{z}+\zeta.$$

Also (96) $$\dot{x}=\dot{\bar{x}}+\dot{\xi},\quad \dot{y}=\dot{\bar{y}}+\dot{\eta},\quad \text{and}\quad \dot{z}=\dot{\bar{z}}+\dot{\zeta}.$$

Hence the kinetic energy of the system,

$$\begin{aligned}\Sigma\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) &= \Sigma\tfrac{1}{2}m\{(\dot{\bar{x}}+\dot{\xi})^2+(\dot{\bar{y}}+\dot{\eta})^2+(\dot{\bar{z}}+\dot{\zeta})^2\}\\ &= \Sigma\tfrac{1}{2}m(\dot{\bar{x}}^2+\dot{\bar{y}}^2+\dot{\bar{z}}^2)+\Sigma\tfrac{1}{2}m(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)\\ &\quad +\dot{\bar{x}}\Sigma m\dot{\xi}+\dot{\bar{y}}\Sigma m\dot{\eta}+\dot{\bar{z}}\Sigma m\dot{\zeta}\\ &= \Sigma\tfrac{1}{2}m(\dot{\bar{x}}^2+\dot{\bar{y}}^2+\dot{\bar{z}}^2)+\Sigma\tfrac{1}{2}m(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2),\end{aligned}$$

since (410) $\Sigma m\dot{\xi} = \Sigma m\dot{\eta} = \Sigma m\dot{\zeta} = 0$. And $\Sigma\frac{1}{2}m(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)$ is the sum of the kinetic energies of the particles moving with velocities equal to their velocities relative to the centre of mass, and $\Sigma\frac{1}{2}m(\dot{\bar{x}}^2+\dot{\bar{y}}^2+\dot{\bar{z}}^2)$ is the kinetic energy of a particle having a mass equal to the mass of the system and a velocity equal to the velocity of the centre of mass.

443. *Examples.*

(1) Two particles of masses m and M moving in a straight line, with velocities v and V respectively ($v > V$) come into collision, the stress between them during collision being in the line of motion, and the co-efficient of restitution being e. Find the loss of kinetic energy.

Let U be the velocity of the centre of mass, which (416) is the same after as before the collision. Then their velocities relative to the centre of mass before the collision are $v-U$ and $V-U$ respectively; and if v' and V' are the velocities of m and M respectively after the collision, $v'-U$ and $V'-U$ are their respective velocities relative to the centre of mass after the collision. Hence (442) the kinetic energy is, before the collision,

$$\tfrac{1}{2}(m+M)U^2+\tfrac{1}{2}m(v-U)^2+\tfrac{1}{2}M(V-U)^2,$$

and, after the collision,

$$\tfrac{1}{2}(m+M)U^2+\tfrac{1}{2}m(v'-U)^2+\tfrac{1}{2}M(V'-U)^2.$$

Hence the loss of kinetic energy is

$$\tfrac{1}{2}m[(v-U)^2-(v'-U)^2]+\tfrac{1}{2}M[(V-U)^2-(V'-U)^2].$$

Now, as both particles have at the instant of collision the velocity U, we have (416),

$$(m+M)U = mv + MV,$$

and (380, Ex. 1),

$$v' = \frac{mv+MV-eM(v-V)}{m+M},$$

$$V' = \frac{mv+MV-em(V-v)}{m+M}.$$

Hence

$$v'-U = -\frac{eM(v-V)}{m+M}$$

$$= e(U-v).$$

And similarly $V' - U = e(U - V)$.

Hence the loss of kinetic energy is

$$\tfrac{1}{2}(1-e^2)[m(v-U)^2 + M(V-U)^2].$$

If therefore $e=1$, there is no loss of energy. If $e=0$, the loss of energy is equal to the energy due to the motion of m and M relative to their centre of mass before the collision.

(2) In any displacement of a system of heavy particles, the work done against the weights of the particles is equal to the product of the weight of the system into the vertical displacement of the centre of mass of the system.

Let m_1, m_2, etc., be the masses of the particles, d_1, d_2, etc., the vertical components of their displacements, x_1, x_2, etc., their initial distances from a horizontal plane. Then the amounts of work done on the various particles are m_1gd_1, m_2gd_2, etc. Hence the whole work done is $g\Sigma md$. Now the vertical displacement of the centre of mass is

$$\frac{\Sigma m(x+d)}{\Sigma m} - \frac{\Sigma mx}{\Sigma m} = \frac{\Sigma md}{\Sigma m}.$$

Hence the product of the weight of the system into this vertical displacement,

$$g\Sigma m \times \frac{\Sigma md}{\Sigma m} = g\Sigma md,$$

which is the whole work done.

(3) Find the work done in raising from the ground the materials (cubical blocks of stone of 1 foot edge and of density 1 cwt. per cubic ft.) in building a uniform column 66 ft. high and 20 ft. square.

Ans. 42,900 foot-tons.

(4) A right pyramid on a square base of 16 ft. side, has an altitude of 24 ft., and stands on a horizontal plane. Find the work necessary to turn it round one of its edges, its density being 100 lbs. per cubic ft.

Ans. 819,200 ft.-pounds.

(5) A chain whose mass is 100 lbs. and length 50 ft. hangs freely by the upper end, which is attached to a drum, upon which the

chain can be wound, the diameter of the drum being so small relatively to the length of the chain that it may be neglected. Find the work done against the weight of the chain in winding up one half of it.

Ans. 1875 ft.-pounds.

(6) The cylindrical shaft of a mine, whose section is 50 sq. ft., contains water (density = 1000 oz. per cubic ft.) to within 90 ft. of the surface. How much will the surface of the water be lowered by an engine working at 10 horse power for 1 hour?

Ans. 54·1 ft.

(7) Find the initial speed of a shot of 1000 lb. mass, discharged from a 100-ton gun, the energy of the charge being 300,000 ft.-pounds, and 1 per cent. being lost in heat, light, etc.

Ans. 24·3... ft. per sec.

444. *Equilibrium of Extended Systems.*—By the equilibrium of a system of particles may be denoted either of two states of motion: (1) a state in which the centre of mass of the system has no linear acceleration, and the system a constant angular momentum about the centre of mass, (2) a state in which the particles of the system are all without linear acceleration. The former may be called a state of molar equilibrium or equilibrium of the system as a whole, the latter a state of molecular equilibrium or equilibrium of the individual particles or molecules of the system.

445. The necessary and sufficient conditions of molar equilibrium may be obtained at once from the equations of 431, viz., $\bar{a} = \Sigma F/\Sigma m$, and $\Sigma FP = \Sigma m(\dot{\overline{\omega r^2}})$. For in order that the acceleration of the centre of mass may be zero, and the angular momentum constant, we must have $\Sigma F = 0$ and $\Sigma FP = 0$; and if these conditions are fulfilled, we have $\bar{a} = 0$ and $\Sigma m(\dot{\overline{\omega r^2}}) = 0$. Hence the necessary and sufficient conditions of molar equilibrium are (1) that the algebraic sum of the components, in any given direction,

of the external forces must be zero, and (2) that the algebraic sum of the moments of the same forces about any axis must be zero also.

446. An expression of the condition of molecular equilibrium may be obtained from the equation of energy (436), which may be written (433)

$$\Sigma\left\{W-\frac{m}{2}(\dot{x}'^2-\dot{x}^2+\dot{y}'^2-\dot{y}^2+\dot{z}'^2-\dot{z}^2)\right\}=0.$$

W is here the work done by all the forces acting on m in any small displacement whose components are $x'-x$, $y'-y$, $z'-z$. ΣW is therefore the work done by all the forces of the system during its corresponding change of configuration. Dividing by t, the time of the small displacement of m, we get

$$\Sigma\left\{\frac{W}{t}-\frac{m}{2}\left(\frac{\dot{x}'-\dot{x}}{t}(\dot{x}'+\dot{x})+\frac{\dot{y}'-\dot{y}}{t}(\dot{y}'+\dot{y})+\frac{\dot{z}'-\dot{z}}{t}(\dot{z}'+\dot{z})\right)\right\}=0.$$

If now the given change of configuration be from one of equilibrium to one indefinitely near it, the component accelerations $(\dot{x}'-\dot{x})/t$, etc., may be put equal to zero. Hence we have $\Sigma(W/t)=0$, *i.e.*, the rate at which the forces acting on the system do work is zero. If the work done by external forces be denoted by w, and that done by internal forces by w', we have $\Sigma W=\Sigma w+\Sigma w'$, and therefore $\Sigma(w/t)=\Sigma(-w'/t)$. Hence if a material system in any given configuration be in molecular equilibrium, the rate at which the external forces do work during any small motion through that configuration is equal to the rate at which work is done against the internal forces; or, if the system is conservative, to the rate of increase of the potential energy of the system due to internal forces.

447. Conversely, if in any small motion of a material system through a given configuration, the rate at which the forces of the system do work is zero, the given configuration is one of molecular equilibrium.

For if not, some of the particles of the system must in that configuration have accelerations. Let them be reduced to equilibrium by the action of forces F_1, F_2, etc., equal to the products of their masses into their accelerations and in directions opposite to these accelerations. Let the system now undergo an indefinitely small change of configuration, such that the particles having accelerations move in the directions of their accelerations. Then work will be done against F_1, F_2, etc., and the rate at which work is done by these forces will in all cases be negative. But the rate at which work is done by all the forces of the system together with F_1, F_2, etc., is zero (446), since the system is now in molecular equilibrium. Hence the rate at which work is done by the forces of the system alone is positive, and cannot be zero. Hence none of the particles of the system can, in the given configuration, have accelerations, and that configuration is therefore one of molecular equilibrium.

448. Hence the necessary and sufficient condition of the molecular equilibrium of a material system in any given configuration is that in any small motion through that configuration the rate at which the external forces do work shall be equal to the rate at which work is done against the internal forces, or, if the system is conservative, to the rate of increase of the potential energy of the system due to internal forces.

449. Hence also the necessary and sufficient condition of the molecular equilibrium of a material system in any given configuration is that in any small motion through that configuration the work done by the external forces shall be equal to that done against the internal forces, or, in other words, that the algebraic sum of the amounts of work done by all the forces shall be zero. In symbols, if F_1, F_2, etc., be the forces acting on the particles of the system, and d_1, d_2, etc., their component displacements in the directions of F_1, F_2, etc., respectively, $\Sigma Fd = 0$.

450. *Stability of Equilibrium.*—If a system of particles which has undergone any indefinitely small change of configuration from that of equilibrium, returns, when left to itself, to the configuration of equilibrium, its equilibrium is said to be *stable* for a change of configuration of that kind. If, when left to itself, the system deviates still more from the configuration of equilibrium, its equilibrium is said to be *unstable.* If the new configuration is also a configuration of equilibrium, the equilibrium of the system is said to be *neutral.* Thus the position of equilibrium of the bob of a pendulum is the lowest point of its swing. If it be slightly displaced from that position and left to itself it will return to that position. Hence its equilibrium is stable. A symmetrical egg may be made to stand on one end; and this position is thus one of equilibrium. But if it be displaced from this position ever so slightly and left to itself, the displacement increases with the time and it falls over on its side. Hence an egg standing on one end is in a position of unstable equilibrium. If a uniform sphere, resting in equilibrium on a horizontal plane, be slightly displaced and left to itself, it will still remain in equilibrium; and thus a uniform sphere on a horizontal plane is in neutral equilibrium.

A configuration of equilibrium of a system may be such that for different small changes of configuration the stability of its equilibrium may be different. Thus a sphere resting on a horizontal cylinder is in neutral equilibrium for small rotations about an axis through the point of contact tangential to the surface and perpendicular to the axis of the cylinder, while, for rotations about all other axes through the same point, its equilibrium is unstable. The equilibrium of a sphere resting in a cylindrical trough is stable for some displacements and neutral for others; and that of a sphere resting on a saddle-back, or *col*, is stable for some displacements and unstable for others. The equilibrium of a system which

is stable, unstable, or neutral, as the case may be, for all possible small displacements of the system is said to be wholly or absolutely stable, unstable, or neutral. The equilibrium of a system which is unstable for any small change of configuration, though it may be stable or neutral for others, is said to be practically unstable.

451. There is a simple relation between the potential energy of a conservative system in its configuration of equilibrium and the stability of its equilibrium. If, after a small displacement from a configuration of equilibrium, the system, when left to itself, returns to the configuration of equilibrium, the forces of the system on the whole do work on the particles of the system in bringing it back to the configuration of equilibrium. Hence, in the configuration of equilibrium, the potential energy of the system is less than in the other configuration. If therefore a system has a configuration in which it is in wholly stable equilibrium, that configuration is one of minimum potential energy. If, after a small displacement from a configuration of equilibrium, the system, when left to itself, deviates still more from the configuration of equilibrium, the forces of the system on the whole do work during the given small displacement, and hence the potential energy of the system is less after the displacement than in the configuration of equilibrium. If therefore a system in a given configuration is in wholly unstable equilibrium, the given configuration is one of maximum potential energy. If, finally, after a small displacement from a configuration of equilibrium a system of particles is still in equilibrium, the forces of the system have neither done work nor had work done against them during the displacement, and hence the potential energy after the displacement is the same as before it.

452. If the potential energy of a system of particles depends wholly upon their weights, the increase of potential energy in any change of configuration (443,

Ex. 2) is the product of the weight of the system into the height through which its centre of mass has been raised. A configuration in which the value of the potential energy is a maximum or minimum, therefore, is one in which the centre of mass has a maximum or minimum height respectively. Hence a configuration of wholly stable or wholly unstable equilibrium is one in which the centre of mass has a lower position or a higher position respectively than in any other configuration into which the system may be brought by an indefinitely small change of configuration. Thus a rod, one end of which is fixed, is in stable equilibrium if the other end, and therefore the centre of mass, is vertically below the fixed end, and is in unstable equilibrium if the other end, and therefore the centre of mass, is above the fixed end.

CHAPTER VI.

DYNAMICS OF RIGID BODIES.

453. A rigid body or system of particles is one whose configuration is invariable, the particles maintaining constant relative positions. Such bodies are purely ideal. But in many cases solid bodies are so slightly deformed by the forces acting on them that for many purposes they may be considered rigid.

It follows from the constancy of the configuration of a rigid body that, if it is rotating about an axis fixed in itself, all its particles must have the same angular velocity, and consequently the same angular acceleration, about that axis; and that the distance of any particle from the axis must be constant. Hence (420) the angular momentum about the given axis, viz., $\Sigma m\omega r^2$, may be written $\omega\Sigma mr^2$, and $\dot{\overline{\omega r^2}}$ becomes $\dot{\omega}r^2$. Hence (225) the rate of change of angular momentum

$$\Sigma m(\dot{\overline{\omega r^2}}) = \dot{\omega}\Sigma mr^2 = \alpha\Sigma mr^2,$$

if α denote the angular acceleration about the given axis.

We found (428) that about an axis fixed in space, $\Sigma FP = \Sigma m(\dot{\overline{\omega r^2}})$. Hence, if α is the angular acceleration about any axis fixed both in space and in the body,

$$\Sigma FP = \alpha\Sigma mr^2.$$

By 430 the same formula applies if a be the angular acceleration about an axis fixed in the body and passing through its centre of mass, whether or not it be fixed also in space.

454. We have thus two equations (414 and 453),

$$\bar{a}=\frac{\Sigma F}{\Sigma m}, \quad a=\frac{\Sigma FP}{\Sigma mr^2},$$

expressing, the one the linear acceleration of the centre of mass, the other the angular acceleration about that point. Hence (251) these equations completely determine the motion of the body.

455. From the second of these equations it follows (Σmr^2 being constant) that the rotating power of a force, or of several forces, about a given axis is proportional to its moment, or to the algebraic sum of their moments respectively, about that axis. This result is frequently assumed by writers on elementary Statics.

456. From the two equations of 454 it follows that a force produces in a rigid body the same kinetic effect at whatever point of its line of action it may be applied. For $\bar{a}$ has the same value, provided the magnitudes and directions of the applied forces are the same; and a has the same value, provided the magnitudes of the applied forces and the distances from the axis of their lines of action are the same. This result is usually called the "*principle of the transmissibility of force*," and is usually made a fundamental hypothesis by writers on Statics.

457. It follows, from the result of 453, that for the complete specification of a force which is acting on a rigid body, it is necessary to know not only its magnitude and direction, as in the case of a particle, but its line of action or some point in its line of action as well.

458. It follows, from the second equation of 454, that if a free rigid body be acted upon by a force whose line of action passes through the centre of mass, it produces in the body no angular acceleration about the centre of mass, and therefore (as is evident from 244) no angular acceleration whatever.

Hence a rigid body initially without rotation, and acted upon only by forces with action lines passing through its centre of mass, can have motion of translation only. The motion of such bodies is discussed in the chapters on the Dynamics of a Particle.

459. *Composition of Forces.*—It is often convenient in investigating the motion of a rigid body to replace the forces acting on it by a simpler set of forces, which would produce the same kinetic effect. Before applying the equations of 454 to the solution of problems, therefore, we will investigate the composition of forces acting on a rigid body, *i.e.*, the reduction of such forces to simpler equivalent systems.

The resultant of the forces acting on a rigid body is the single force or the simplest system of forces which will produce in it the same accelerations as are produced by the given forces.

460. Any coplanar forces acting on a rigid body are reducible to a single force.—A force F, whose components in rectangular directions in the plane of the forces are F_x, F_y, will produce the same linear acceleration of the centre of mass as the acting forces (components X_1, Y_1, X_2, Y_2, etc.), provided $F_x = \Sigma X$ and $F_y = \Sigma Y$; and it will produce the same angular acceleration about any point O in the plane of the component forces if its line of action is at such a distance (p) from the point O that Fp is equal to the algebraic sum (N) of the moments of the

forces about it,* if therefore $Fp=N$. Hence, as $F=(F_x^2+F_y^2)^{\frac{1}{2}}$, the forces are reducible to a single force if

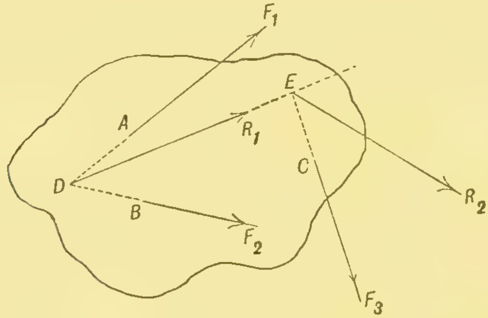
$$p[(\Sigma X)^2+(\Sigma Y)^2]^{\frac{1}{2}}=N.$$

A finite value of p can always be found to satisfy this equation, provided $[(\Sigma X)^2+(\Sigma Y)^2]^{\frac{1}{2}}$ is not equal to zero. Hence, except in this case, a single resultant can always be found.

461. *Determination of the Single Resultant of Coplanar Systems of Forces.*—The forces may or may not be parallel.

Case I.—Non-parallel Coplanar Forces.—Analytical Determination.—It follows, from 460, that the magnitude of the single resultant is $[(\Sigma X)^2+(\Sigma Y)^2]^{\frac{1}{2}}$, its direction cosines $(\Sigma X)/F$, $(\Sigma Y)/F$, and its distance from the point O N/F. The magnitude and direction of the resultant force are thus the same as if forces of the same magnitude and direction as the given forces acted upon a particle (313 and 90).

462. *Geometrical Determination.*—The magnitude and line of action of the resultant may also be found by the



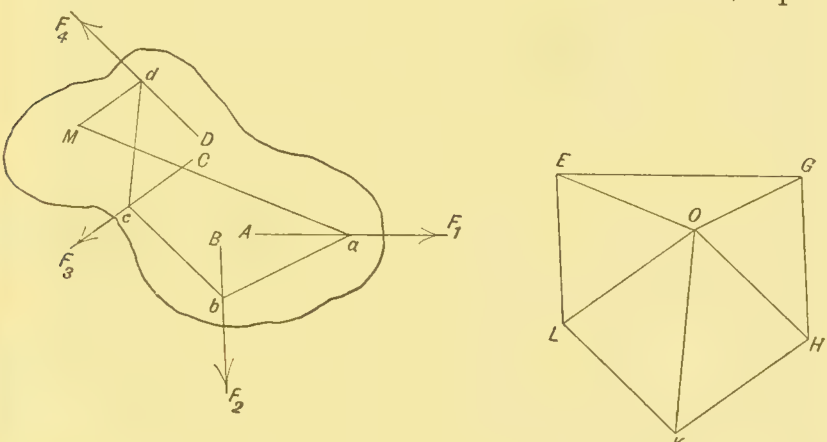
aid of 456.—Let coplanar forces F_1, F_2, F_3 act on a rigid body at A, B, C respectively. Produce F_1 and F_2 till

* The moment of a force about a point is its moment about an axis through the point perpendicular to the plane containing the point and the line of action of the force.

they meet in D. Let both forces (456) act at D instead of at A and B, and let R_1, their resultant, be determined by 313. Produce R_1 to meet F_3 in E. Let now R_1 and F_3 act at E instead of C and D, and let R_2, their resultant, be determined. Then R_2 is the resultant of the given forces.

By thus applying the parallelogram law the resultant may be determined either by calculation or graphically (382, Ex. 22).

463. The following is a more elegant graphical method: Let forces F_1, F_2, F_3, F_4 act as represented in the diagram at the points A, B, C, D. From any point E draw EG, from G draw GH, from H HK, and from K KL, repre-



senting in magnitude and direction the forces F_1, F_2, F_3, F_4 respectively. Then (461) EL represents their resultant in magnitude and direction. To find a point in its line of action, take any point O and join it to E, G, H, K, L. From any point in F_1, say a, draw a line parallel to OG and meeting F_2 in b. From b draw a line parallel to OH, meeting F_3 in c. From c draw a line parallel to OK, meeting F_4 in d. From d draw a line parallel to OL, and from a a line parallel to OE, and let them meet in M.

A force represented by EG may be resolved into two represented by EO and OG. Hence F_1 is equivalent to forces proportional to EO and OG, with lines of action Ma and ba. Similarly F_2 may be resolved into forces proportional to GO and OH, with lines of action ab and cb, F_3 into forces proportional to HO and OK, with lines of action bc and dc, and F_4 into forces proportional to KO and OL, and with lines of action cd and dM. Hence the given system of forces is equivalent to single forces in the lines dM and Ma, and pairs of equal and opposite forces in each of the lines ab, bc, cd. The resultant of this system is clearly a force through M. Hence the required resultant is a force represented by EL and acting at M.

The above construction by which the line of action is determined is frequently called the funicular polygon or link polygon construction. Its application to engineering problems will be found discussed in books on Applied Mechanics.

464. *Case II. Parallel Coplanar Forces.*—If the given forces are parallel, the constructions of 462 and 463 fail. In any such case, however, a system equivalent to the given system may be obtained by introducing two equal and opposite forces in the same line, and with directions inclined to those of the given parallel forces; and to this equivalent system the above constructions may be applied.

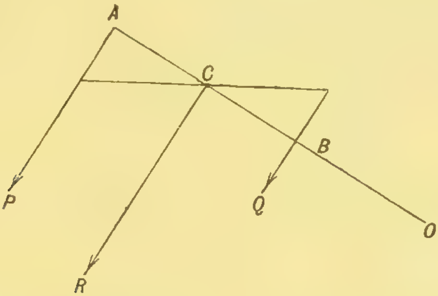
465. We may find the resultant of parallel forces more readily, however, as follows:—

First, let there be two such component forces. These may be either codirectional or opposite in direction.

(*a*) *The Forces Codirectional.*—Let P and Q be forces acting in the same direction on a rigid body of mass m. Then, that the resultant R may produce in the centre of mass of the body the same acceleration as P and Q, its

line of action must be parallel to theirs, and we must have

$$\bar{a}=(P+Q)/m=R/m,$$



and hence $R=P+Q$. Also that R may produce about any point O, the same angular acceleration, as P and Q, its moment about O must be equal to the algebraic sum of their moments. From O draw OBA perpendicular to P and Q, and therefore to R, and meeting P, Q, and R in A, B, and C respectively. Then

$$P\,.\,AO+Q\,.\,BO=R\,.\,CO=(P+Q)CO.$$

It follows that CO is intermediate in length between AO and BO, and that C is therefore between A and B. Substituting for AO and BO their values we have

$$P(AC+CO)+Q(CO-CB)=(P+Q)CO.$$

Hence
$$P\,.\,AC=Q\,.\,CB,$$

i.e., R's line of action cuts the line AB (and, therefore, any line intersecting P and Q), so that the products of the forces into the segments adjacent to them are equal.

466. (*b*) *The Forces Opposite in direction and Unequal.*—Let P and Q be the given forces. Then, as above, if P be greater than Q, $R=P-Q$ and is codirectional with P, and

$$P\,.\,AO-Q\,.\,BO=R\,.\,CO=(P-Q)CO.$$

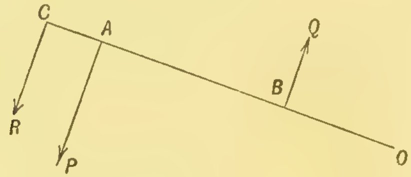
Now BO is less than AO. Hence

$$(P-Q)AO<P\,.\,AO-Q\,.\,BO\;;$$

and therefore AO is less than CO, and C is a point in BA produced. Substituting in the above equation the values of AO and BO, we have

$$P(CO-CA)-Q(CO-CB)=(P-Q)CO,$$

and
$$P\,.\,CA=Q\,.\,CB;$$



i.e., R's line of action cuts the line BA produced, so that the products of the component forces into the segments adjacent to them are equal.

467. (*c*) *The Forces Opposite and Equal.*—A system of two forces equal and opposite, but not in the same straight line, is called a *couple* or a *torque.*

In this case
$$R=0,$$

and
$$P\,.\,AO-Q\,.\,BO=P\,.\,AB=0\times CO.$$

Now $P\,.\,AB$ has a finite value. Hence CO must be infinitely great. The single resultant of two equal and opposite parallel forces is therefore a force zero at an infinite distance. In other words, a couple can produce rotational, but not translational, acceleration in the body on which it acts.

As $P\,.\,AB$ has the same value for all positions of O, the moment of a couple about all points in its plane, and therefore about all axes perpendicular to its plane, is the same, and is equal to the product of either force into the distance between their lines of action. This distance is called the *arm* of the couple.

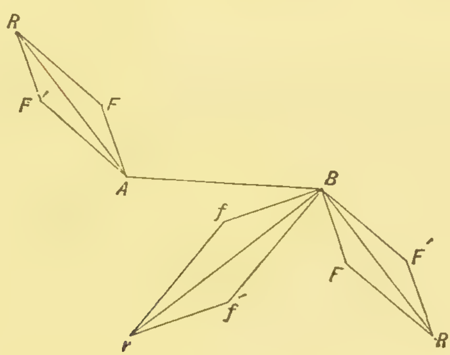
A couple is therefore completely specified if its moment and the direction of a line perpendicular to its plane are given. It may therefore be represented by a straight

line, whose length is proportional to the magnitude of the moment of the couple, whose direction is normal to the plane of the couple, and which is so drawn, according to a convention similar to that of 103, as to indicate the sign of the moment. Such a line is usually called the *axis* of the couple.

468. It follows from the above that all couples which have equal moments of the same sign, and are in the same or in parallel planes, produce the same kinetic effect, or are equivalent, whatever may be their length of arm or the magnitudes or lines of action of their forces.

469. *Composition of Couples.*—The resultant of any number of component couples is a couple, and is to be determined by the parallelogram law.

Let the planes of two component couples intersect in the line AB. At A and B let equal and opposite forces, F, act in the plane of one of the component couples at



right angles to AB, and of such magnitude and direction that the couple $F \,.\, AB$ has the same moment and sign as the component couple in its plane. At A and B let equal and opposite forces F' act in the plane of the second component couple, and at right angles to AB, F' being of such magnitude and direction that the couple $F' \,.\, AB$ has the same moment and sign as the second component

couple. Then the couples $F.AB$ and $F'.AB$ are equivalent to the two component couples. Let AF and BF, AF' and BF', represent the forces F and F'. Then if the parallelograms $AFRF'$, $BFRF'$, be completed, the diagonals AR and BR will represent the resultants of F and F' at A and B respectively. Since the angles FAF' and FBF' are equal, the parallelograms FF' are similar. Hence the angles FAR and FBR are equal, and therefore the equal resultants, R, are in the same plane. Since in each case R is in the same plane as F and F', AR and BR are perpendicular to AB. Hence the two component couples are equivalent to the couple $R.AB$.

From B draw Bf, Bf' and Br, the axes of the couples $F.AB$, $F'.AB$, and $R.AB$ respectively. Bf, Bf', and Br are thus perpendicular to the planes $FABF$, $F'ABF'$, and $RABR$ respectively; and consequently the angles fBr, $f'Br$ are equal to FBR, $F'BR$ respectively. Also, since the couples represented by Bf, Bf', and Br have the same arm, we have

$$Bf : BF = Bf' : BF' = Br : BR.$$

Hence, if r be joined to f' and f, $rfBf'$ will be a parallelogram; and consequently the axis of the resultant couple is to be determined from the axes of the component couples by the parallelogram law (78).

If there are more than two component couples, the resultant of any two may be compounded with a third, their resultant with a fourth, and so on until the resultant of all has been found.

It follows that the laws of the resolution of couples are the same as in the case of displacements, velocities, etc.

470. Secondly (465), let there be any number of component parallel forces. In that case the resultant of any two may first be determined, then the resultant of their resultant, and a third, and so on, until the resultant of all has been found.

471. Any system of parallel forces, whether coplanar or not, may be reduced to a single force.—For, as any two parallel lines are necessarily in the same plane, the resultant of any two of the given forces is coplanar with a third, that of any three with a fourth, and so on. Thus the single resultant of a non-coplanar system may be determined as in 470.

472. From 471, 465, and 399 it is clear that the above process is exactly that by which the centre of mass of a system of particles was determined, the magnitudes of the parallel forces taking the place of the masses of the particles, and the positions of their points of application that of the positions of the particles. Hence, as in 400 it may be shown that if F_1, F_2, etc., are the magnitudes of the parallel forces, and d_1, d_2, etc., the distances of their points of application from any given plane, the distance from it of the point of application of their resultant is $\Sigma Fd/\Sigma F$. The point of application of the resultant is called the *centre* of the system of parallel forces.

473. In the special case in which all the particles of a body are acted upon by parallel forces proportional to their masses, the centre of parallel forces is an important point. If F_1, F_2, etc., are the parallel forces, and m_1, m_2, etc., the masses of the particles on which they act, $F_1 = km_1$, $F_2 = km_2$, etc., where k is a constant. Hence the distance of the centre of the system of parallel forces from any plane from which the distances of the particles are d_1, d_2, etc., is

$$\frac{\Sigma kmd}{\Sigma km} = \frac{k\Sigma md}{k\Sigma m} = \frac{\Sigma md}{\Sigma m}.$$

And this is the distance of the centre of mass. Hence the centre of the above system of parallel forces coincides with the centre of mass.

474. If a body be sufficiently small relatively to the earth, the weights of its particles may be considered to be parallel forces; and they are proportional to the masses of the particles, for they produce in the particles the same acceleration, g. Hence the weights of the particles of a sufficiently small body are reducible to a single force equal to g times the mass of the body and acting vertically downwards through the centre of mass, whatever the position of the body may be. For this reason the centre of mass is often called the *centre of gravity*.

The term *centre of gravity* has also the following signification to which it should be restricted: If a body attracts and is attracted by all external bodies, whatever their distance and relative position, as though its mass were concentrated in a point fixed relatively to it, that point is called its centre of gravity, and the body is said to be centrobaric or barycentric. In general, bodies are not centrobaric. We have seen (316, Ex. 6) that a uniform sphere or spherical shell has this property.

If a body has a centre of gravity it necessarily coincides with the centre of mass. For, as we have seen (473), the resultant attraction of an infinitely distant body, whose attractions on its particles would be parallel forces, would pass through the centre of mass whatever the position of the body.

475. *Examples.*

(1) Three forces act at the middle points of the sides of a rigid triangular plate, in its plane, each force being perpendicular and proportional to the side at which it acts. If the forces are all inwards or all outwards, the resultant is zero.

(2) If a rigid plane quadrilateral $ABCD$ be acted upon by four forces, represented in magnitude, direction, and line of action by AB, CB, AD, CD respectively, the line of action of the resultant will be the line joining the middle points of the diagonals; and its

magnitude will be represented by four times the length of that line.

(3) A system of any number of coplanar forces being represented by the several sides of a closed polygon, as described by the continued motion of a point in a plane, show that the sum of their moments round any point in the plane is independent of the position of the point.

(4) If six forces acting on a rigid body be completely represented, three by the sides of a triangle taken the same way round, and three by the sides of the triangle formed by joining the middle points of the sides of the original triangle, and if the parallel forces act in the same direction, and the scale on which the first three forces are represented be four times as large as that on which the last three are represented, the given six forces produce neither translational nor rotational acceleration.

(5) Forces of 10, 20, 30, and 40 poundals act on a rigid body at A, B, C, D, the corners of a square whose side is 2 feet, and in its plane. Their inclinations to AB, BC, CD, DA are 45°, 90°, 30°, 60° respectively. Show that their resultant is a force of 35·65... poundals, and that its line of action is distant 3·03... ft. from C.

(6) Parallel forces in the same direction, and of the magnitudes 10, 15, 20, 25, act at points A, B, C, D respectively of a straight rod, the distances AB, BC, CD being 2, 3, and 4 respectively. Find the distance of the point of application of the resultant from A.

Ans. 5·07....

(7) Two parallel forces in opposite directions, and of magnitudes 20 and 5, act at points A and B respectively of a rigid body 4 feet apart. Find the distances from A and B of the point in which their resultant line of action cuts AB.

Ans. $1\frac{1}{3}$ and $5\frac{1}{3}$ ft.

(8) At each end of each side of a uniform triangular plate a force acts parallel and proportional to the line drawn from the opposite vertex to bisect that side. Show that the resultant of the six forces passes through the centre of mass of the triangle.

(9) A triangular lamina ABC at rest is moveable in its own plane about a point in itself. Forces act on it along and proportional to BC, CA, BA. Show that if they do not move the lamina, the point must lie in the straight line bisecting BC and CA, and that the reaction at the point is proportional to $2AB$.

(10) Two parallel forces in the same direction and proportional to two of the sides of a triangle, act at the angles of the triangle, opposite the sides to which they are proportional respectively. Show that their resultant passes through the point in the third side in which it is cut by the line bisecting the opposite angle.

(11) The numerical measures of the magnitude of a force which acts at a point in a given direction, and of the distances of the point from two straight lines at right angles to one another in the same plane with it, are denoted by a, b, c; but it is not known which is which. Find the centre of all the forces which may be represented.

Ans. Distance from each line $=\dfrac{ab+bc+ca}{a+b+c}$.

(12) Forces 1, -3, -5, 7 act on a rigid rod at points A, B, C, D, whose distances are such that $AB=3$, $BC=2$, $CD=2$. Find the magnitude of the resultant couple.

Ans. 15.

(13) Three equal and codirectional forces (F) act at three corners of a square (side $=a$) perpendicularly to the square. Find (a) the magnitude of the force which, applied at the other corner of the square, would with the given forces constitute a couple, and (b) the moment of the couple.

Ans. (a) $3F$; (b) $Fa2\sqrt{2}$.

(14) ABC is a triangle right-angled at B. At A a force F is applied in the plane of the triangle perpendicular to AC; at C a force $2F$ in the same direction; and at B a force $3F$ in the opposite direction. Find the moment of the resulting couple.

Ans. $F(AB^2-2BC^2)/AC$.

(15) The resultant of three forces represented by the sides of a triangle taken the same way round is a couple whose moment is proportional to the area of the triangle.

476. Any forces whatever acting on a rigid body are reducible to a system of two forces.

A force, F, whose components in the directions of rectangular axes are F_x, F_y, F_z, acting at any chosen point whose distances from the centre of mass of the body in the directions of the components are ξ, η, ζ, will produce the angular acceleration about the centre of mass produced by the acting forces, provided (427)

$$F_z\eta - F_y\zeta = L,\ F_x\zeta - F_z\xi = M,\ F_y\xi - F_x\eta = N,$$

where L, M, N are the algebraic sums of the moments of the acting forces about axes through the centre of mass, parallel to the x, y, z axes respectively. We may reduce these equations to one by multiplying the first by F_x, the second by F_y, and the third by F_z, and adding, by which process we find that

$$LF_x + MF_y + NF_z = 0$$

is the condition which must be satisfied that the force F may produce the required angular acceleration. It is obvious that values of F_x, F_y, F_z can always be found to satisfy this equation. These values will be different for different chosen points of application.

The force F with another force F', whose components are F_x', F_y', F_z', and which acts at the centre of mass, will produce the linear acceleration produced by the acting forces, provided

$$F_x + F_x' = \Sigma X,\ F_y + F_y' = \Sigma Y,\ F_z + F_z' = \Sigma Z,$$

where ΣX, ΣY, ΣZ are the sums of the components of the acting forces in the directions of the x, y, z axes respectively. As F' acts at the centre of mass it has (458) no effect on the body's angular acceleration.

Now, whatever may be the values of F_x, F_y, F_z which satisfy the first condition, values of F_x', F_y', F_z' may be

found to satisfy the last three equations. Hence any forces acting on a rigid body are reducible to two forces. As the point ξ, η, ζ chosen above was any point whatever, the forces acting on a rigid body may be reduced to any one of an infinite number of pairs of forces.

477. To determine the condition of the reducibility of a system of forces acting on a rigid body to a single force.

As this force must produce both the linear and the angular accelerations produced by the acting forces, we have, if F_x, F_y, F_z are its rectangular components, and ξ, η, ζ the co-ordinates, relative to the centre of mass, of its point of application, and if L, M, N are the moments of the acting forces about axes, parallel to the axes of co-ordinates, through the centre of mass,

$$F_x = \Sigma X,\ F_y = \Sigma Y,\ F_z = \Sigma Z;$$

and $$F_z\eta - F_y\zeta = L,\ F_x\zeta - F_z\xi = M,\ F_y\xi - F_x\eta = N.$$

These six equations may, as in 476, be reduced to the single equation

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0,$$

which therefore is the condition which must be fulfilled that the resultant of the given forces may be a single force.

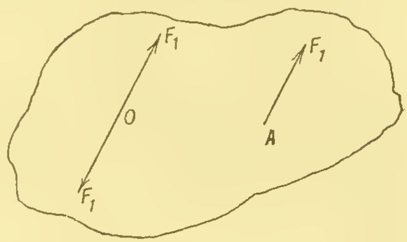
478. The magnitude of this resultant force is clearly

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

Its direction cosines are $\Sigma X/R$, $\Sigma Y/R$, $\Sigma Z/R$. If we put $\xi = 0$ in the equations of 477, we obtain $\zeta = M/F_x$, $\eta = -N/F_x$. These therefore are the co-ordinates of the point in which the line of action of the force cuts the $\eta\zeta$ plane.

479. Any forces acting on a rigid body may be reduced to a single force and a single couple.

If F_1 be any one of the forces acting on a rigid body, there may be introduced at any point O, without any change of the motion of the body, a pair of equal and opposite forces, F_1, parallel to the original F_1; and for every force acting on the body we thus obtain an equal force in the same direction acting at O, and a couple (called the couple of transference). The forces at O give a resultant force at O, and the couples compound into a resultant couple (469).



Whatever point O may be chosen, the direction and magnitude of the resultant force will clearly be the same. The resultant couple will however be different for different positions of O.

480. To determine the resultant force and couple for any given system of forces and for any given position of O.

Let $X_1, Y_1, Z_1, X_2, Y_2, Z_2$, etc., be the components of the forces of the system in the directions of rectangular axes through O, and let $x_1, y_1, z_1, x_2, y_2, z_2$, etc., be the co-ordinates of their respective points of application. Then as the resultant force R is the same for all positions of O, it must be the same as the force which at the centre of mass would produce the linear acceleration produced by the system of forces. Hence

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2};$$

and its direction cosines are $\Sigma X/R$, $\Sigma Y/R$, $\Sigma Z/R$. As the component couples must produce about the chosen axes the same angular accelerations as the forces of the system, they must be equal to the moments of the forces about these axes. Hence if L, M, N are the component

couples whose axes have the directions of the x, y, z axes respectively,

$$L=\Sigma(Zy-Yz),\quad M=\Sigma(Xz-Zx),\quad N=\Sigma(Yx-Xy).$$

Hence (469 and 88) the resultant couple

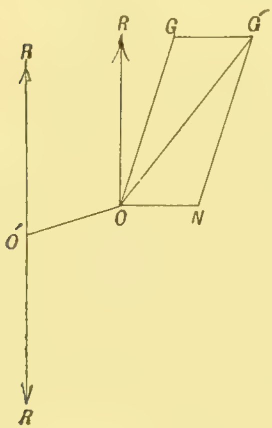
$$G=\sqrt{L^2+M^2+N^2}$$
$$=\sqrt{[\Sigma(Zy-Yz)]^2+[\Sigma(Xz-Zx)]^2+[\Sigma(Yx-Xy)]^2},$$

and its direction cosines are L/G, M/G, N/G. Hence also (8) the inclination of the axis of the resultant couple to the resultant force is

$$\cos^{-1}\left(\frac{\Sigma X}{R}\cdot\frac{L}{G}+\frac{\Sigma Y}{R}\cdot\frac{M}{G}+\frac{\Sigma Z}{R}\cdot\frac{N}{G}\right).$$

481. The resultant force being given, and the resultant couple for a given point of application of the resultant force, to find the resultant couple for any other point of application.

Let OR and OG represent the resultant force and couple when the resultant force acts at O. Let O' be the other point of application. At O' introduce two opposite forces R equal and parallel to the force R at O. They will not affect the motion of the body. Now the forces R at O and O' constitute a couple, whose axis ON is perpendicular to the plane of $ROO'R$, and is proportional to the product of R into the perpendicular distance of OR from $O'R$. The two couples OG and ON give (469) a resultant couple OG'. Hence the given system of forces is equivalent to a force R acting at O' and a couple OG'.

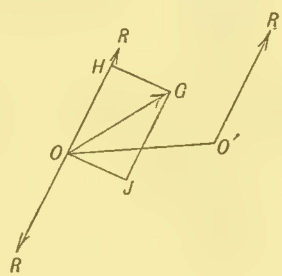


If O' is in the line of action of OR, it is evident that ON is zero and that OG' is the same as OG. If O' is anywhere else, ON will have a value and OG' will differ from

OG. It is obvious that any other line of action of the resultant force than that through O' must either be at a greater or smaller distance from OR than $O'R$, or be so placed that the plane through it and OR is inclined to the plane $ROO'R$, and that therefore ON, and consequently also OG', can have a given magnitude and direction only for one line of action of the resultant force. Hence in a given body, acted on by given forces, there is but one line, such that, if the resultant force acts in it, the resultant couple will have a given magnitude and direction.

482. Any forces acting on a rigid body may be reduced to a single force and a couple whose axis is parallel to the line of action of the force.

Let R be the force (OR being its line of action) and G the couple (OG being its axis), which together form the resultant of the acting forces according to 479. The couple OG may be resolved into two, whose axes are OH and OJ, in and perpendicular to the direction of OR respectively. The couple OJ is in the plane through OR perpendicular to the plane of OR and OG. Let OO',



drawn perpendicular to the plane of OR and OG, be the length of arm of the component couple OJ when its forces are made equal to R, and let the forces R of this couple act at O and O' in directions perpendicular to OO'. Then we have two forces, R, acting at O in opposite directions. Hence the original force R, together with the component couple OJ, are equivalent to a force R at O', having the same direction as the original R. Hence the given system, viz., the force R acting at O and the couple OG, is reduced to the force R acting at O' and a couple whose axis OH is parallel to R.

When this reduction is made, the line of action of the force is called the *central axis* of the system of forces,

and as this theorem is due to Poinsot, it is usually called Poinsot's central axis. It follows from 481 that a given system of forces can have but one central axis.

Sir R. S. Ball has given the name *wrench* to the resultant force and couple to which a given system of forces may be reduced when the line of action of the resultant force is the central axis.

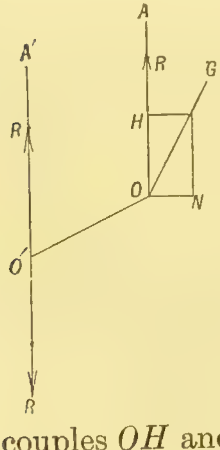
483. If the angle ROG (482) is θ, the component couples OH and OJ are $G\cos\theta$ and $G\sin\theta$ respectively. Hence

$$OO' = (G\sin\theta)/R.$$

If the direction cosines of OR and OG are l, m, and n, λ, μ, and ν respectively, those of OO' will be (10) $(m\nu - n\mu)/\sin\theta$, $(n\lambda - l\nu)/\sin\theta$, $(l\mu - m\lambda)/\sin\theta$. Now the products of OO' into its direction cosines are the coordinates of O' relative to O. Hence, employing the values of l, m, n, λ, μ, ν found in 480, we obtain as coordinates of O'

$$(N\Sigma Y - M\Sigma Z)/R^2,\ (L\Sigma Z - N\Sigma X)/R^2,\ (M\Sigma X - L\Sigma Y)/R^2.$$

The direction of the central axis being thus known, and the position of one point in it, the axis is completely determined.



484. The magnitude of the resultant couple is less when its direction is that of the central axis than when it has any other direction.

Let OA be the central axis of a given system of forces, $O'A'$ any parallel line, and OO' a line perpendicular to both. Let R, acting at O, be the resultant force, and OH the resultant couple. At O' introduce two opposite forces, equal and parallel to R. Then the system is equivalent to a force R acting at O' and the couples OH and ON, ON being perpendicular to the plane $ROO'R$ and therefore to OH, and proportional to the

product of R into OO'. The resultant of these couples is one represented by OG, which is necessarily greater than OH.

485. *Examples.*

(1) The magnitudes, directions, and lines of action of four forces acting on a rigid body are represented by four sides of a skew quadrilateral taken the same way round. Show that the system is equivalent to a couple whose axis is perpendicular to both diagonals.

(2) Show, by using the result of 477, that the resultant of any system of parallel forces is a single force.

(3) $ABCD$ is a tetrahedron, the angles BAC, CAD, DAB being right angles. At the centres of mass of the faces BAC, CAD, DAB forces act (all inwards or all outwards), with directions perpendicular to the faces, and magnitudes proportional to the areas of the faces. Show that their resultant is a single force.

(4) When a force is transferred to any point O, the resolved part of the couple of transference in any direction OZ is equal to the moment of the given force about OZ.

(5) OA, OB, OC are conterminous edges of a cube and CD, EF are edges parallel to OB and OC respectively. Find the distance from O of the central axis of a system of three equal forces completely represented by OA, CD, and EF.

Ans. $AC/3$.

(6) OA, OB, OC are conterminous edges of a rectangular parallelopiped, so related that a positive rotation of 90° about OA as axis would bring OB to the initial position of OC. Forces proportional to OA, OB, OC (whose lengths are a, b, c respectively) act at B, C, and A in the directions OA, OB, OC respectively. Find the central axis.

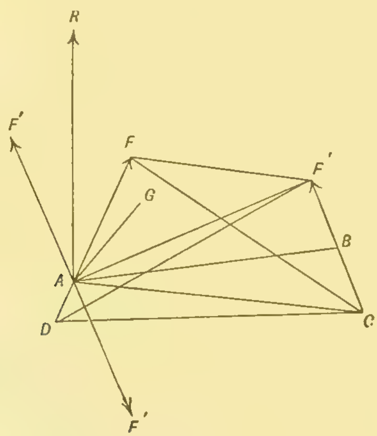
Ans. Its direction is that of the diagonal through O, and it passes through a point whose distances from the planes BC, AC, and AB are respectively

$$\frac{ac^2-ab^2}{a^2+b^2+c^2},\quad \frac{a^2b-bc^2}{a^2+b^2+c^2},\quad \frac{b^2c-a^2c}{a^2+b^2+c^2}.$$

(7) A system of forces can always be reduced to two forces whose lines of action are at right angles to one another. [Through a point P in the central axis draw a straight line, OPO', perpendicular to the central axis, bisected in P, and of such length that if F is the magnitude of the resultant force when its line of action is the central axis, $F.OP$ is equal to the resultant couple. Resolve the resultant force, F, into two equal and parallel forces, $F/2$, at O and O', and let the forces of the resultant couple act at O and O' also.]

(8) The volume of the tetrahedron, opposite edges of which represent the forces of any one of the infinite number of pairs of forces to which a given system of forces may be reduced, is constant.

Let F and F' be any such pair of forces, and let DF, CF' represent them, $DFCF'$ being therefore the tetrahedron referred to in the problem. Let AB be a line perpendicular to both DF and CF'.



At A introduce two opposite forces F' equal and parallel to CF'. Let R be the resultant of F and the force F' which acts at A and is codirectional with CF'. Draw AG perpendicular to the plane of the forces F' and representing the couple whose arm is AB and whose forces are the forces F' acting at C and A. Then the force AR and the couple AG form the resultant of the given system. DF, AF'', and AG being all perpendicular to AB are in the same plane and AG is perpendicular to AF''. The resultant couple H

whose direction is that of the central axis is (482, 483) such that

$$H = AG \cos RAG$$
$$= CF' \,.\, AB \,.\, \sin RAF'$$
$$= CF' \,.\, AB \,.\, \sin FAF' \,.\, DF/R.$$

Hence $$H \,.\, R = CF' \,.\, DF \,.\, AB \,.\, \sin FAF'.$$

Now (481) $H \,.\, R$ is constant. And $AB \,.\, CF'$ being equal to twice the area of the triangle ACF', and $DF \sin FAF'$ being the length of the projection of DF on a line perpendicular to the plane of the same triangle, $CF' \,.\, DF \,.\, AB \,.\, \sin FAF'$ is equal to six times the volume of the tetrahedron $FDCF'$. Hence the volume of this tetrahedron is constant.

486. *Moments of Inertia.*—The quantity Σmr^2 (453), the sum of the products of the masses of the particles of a rigid body into the squares of their distances from a fixed axis in it, is called the moment of inertia of the body about the given axis.

If M is the mass of the body, a quantity k can always be found such that $Mk^2 = \Sigma mr^2$. The quantity k thus found is called the *radius of gyration* of the body about the given axis.

Moments of inertia may be determined either by experiment or by calculation.

487. *Determination by Experiment.*—Let the body whose moment of inertia I about a given axis is to be determined be so mounted that the given axis is fixed. Let it then be acted upon by a known force F at a known distance p from the axis, and in a plane perpendicular to the axis, and let the angular acceleration a be observed. We have then (453) $aI = Fp$ and $I = Fp/a$.

It is difficult thus to apply a known force to a body at a known distance from a given axis in it and to observe the angular acceleration. But it is generally easy to apply the same force or set of forces at the same distance

or distances in successive experiments. Hence a moment of inertia is more readily determined by two experiments than by one. First, let the angular acceleration of the body under investigation be observed when under the action of forces whose moment ΣFp is constant from experiment to experiment. We have as above $aI = \Sigma Fp$. Next, let a body, whose moment of inertia (I') about a given axis is known, be rigidly attached to the given body so that the axis about which its moment of inertia is known is in the same straight line with the fixed axis of the given body; and let the same forces be applied in the same way as before. If the angular acceleration is now found to be a', we have $a'(I+I') = \Sigma Fp$. Hence $aI = a'(I+I')$, and

$$I = \frac{a'I'}{a-a'}.$$

It is practically impossible to observe the angular acceleration. But the forces employed may readily be so applied (see 588) that the sum of their moments may be directly proportional to the angular displacement (θ) of the body, and that they may tend always to bring the body to a position in which its angular displacement is zero. In that case the body will oscillate, the ratio of its angular acceleration to its angular displacement will be independent of its angular displacement, and every point of the body will therefore have a simple harmonic motion. Hence (163) the time of oscillation will be $t = 2\pi\sqrt{\theta/a}$, and will be independent of the extent of the oscillation. For any given value of θ therefore $a \propto 1/t^2$. Hence, the times of oscillation in the above experiments having been observed to be t and t' respectively, we have

$$I = \frac{t^2 I'}{t'^2 - t^2}.$$

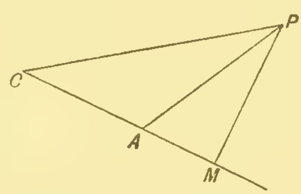
The best methods of applying the force and of observing the times of oscillation will be found described in books on Laboratory Practice.

488. *Determination by Calculation.*—To effect the summation indicated by the formula Σmr^2 the Integral Calculus is in general necessary. But in the case of bodies of simple geometrical form and of uniform density the summation may be effected by elementary mathematical methods.

In the determination by calculation the following propositions will be found useful:

(1) The moment of inertia of a body about a given axis is equal to the moment of inertia of the body about a parallel axis through the centre of mass, together with the product of the mass of the body into the square of the distance between the two axes.

Let P be the position of any particle of the body of mass m. Let the plane of the diagram intersect the given axis and the parallel axis through the centre of mass, normally in A and C respectively. Let d be the distance between the axes, s the distance from the axis C of the foot M of the perpendicular PM from P on CA or CA produced, *i.e.*, the distance of P from a plane through the centre of mass and perpendicular to CA. Let the length of PM be p. The moment of inertia of the body about A,



$$\begin{aligned}\Sigma m \,.\, AP^2 &= \Sigma m[(s-d)^2+p^2],\\ &= \Sigma m(s^2+p^2)+d^2\Sigma m - 2d\Sigma ms,\\ &= \Sigma m \,.\, CP^2 + d^2\Sigma m,\end{aligned}$$

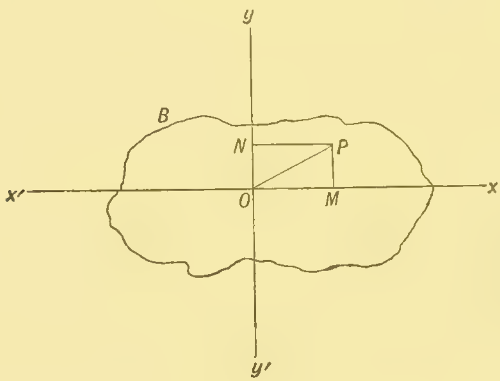
since (403) $\Sigma ms = 0$. Now $\Sigma m \,.\, CP^2$ is the moment of inertia of the given body about an axis parallel to the given axis, through the centre of mass; and $d^2\Sigma m$ is the product of the mass of the body into the square of the distance between the axes.

If M is the mass of the body, and k its radius of gyration about an axis in the given direction through the

centre of mass, the moment of inertia about the given axis is $M(k^2+d^2)$.

489. (2) The moment of inertia of a plane lamina about an axis perpendicular to it through a given point is equal to the sum of its moments of inertia about axes in its plane through the given point, and perpendicular to one another.

Let xx' and yy' be perpendicular axes through the point O in the plane of the lamina B. Let P be the position of any particle of mass m; PM, PN, PO its distances from xx', yy', and an axis through O perpendicular



to xx' and yy'. Then the moments of inertia of m about these axes are $m \cdot PM^2$, $m \cdot PN^2$, $m \cdot PO^2$ respectively, and those of the lamina are thus $\Sigma m \cdot PM^2$, $\Sigma m \cdot PN^2$, $\Sigma m \cdot PO^2$ respectively. Now $PO^2 = PM^2 + PN^2$. Hence

$$m \cdot PO^2 = m \cdot PM^2 + m \cdot PN^2$$

and $$\Sigma m PO^2 = \Sigma m PM^2 + \Sigma m PN^2.$$

490. *Examples.*

(1) Find the moment of inertia of a uniform thin straight rod (length $= l$, mass $= M$) about an axis perpendicular to its length, (a) through one end point, and (b) through its middle point.

(a) Let the rod be divided into an indefinitely great number (n) of equal parts (length $= a$). The distance of the middle point of

each of these parts from the axis may be taken to be the distance of the part itself. Let ρ be the linear density. Then if I denote the moment of inertia about the given axis,

$$I=\rho a\left(\frac{a}{2}\right)^2+\rho a\left(\frac{3a}{2}\right)^2+\rho a\left(\frac{5a}{2}\right)^2+\text{etc.}+\rho a\left(\frac{(2n-1)a}{2}\right)^2$$

$$=\rho\frac{a^3}{4}[1+3^2+5^2+\text{etc.}+(2n-1)^2]$$

$$=\rho\frac{a^3}{4}\cdot\frac{4}{3}\left(n^3-\frac{n}{4}\right)$$

$$=\rho\left(\frac{l^3}{3}-\frac{la^2}{12}\right)$$

$$=\rho\frac{l^3}{3}, \text{ since } a \text{ is indefinitely small,}$$

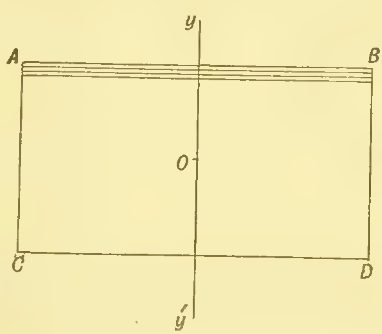
$$=M\frac{l^2}{3}.$$

If k is the radius of gyration, $k=l/\sqrt{3}$.

(b) The moment of inertia of each half about its end point is by Ex. 1 (a),
$$\frac{M}{2}\cdot\left(\frac{l}{2}\right)^2\cdot\tfrac{1}{3}=M\frac{l^2}{24}.$$
Hence the moment of inertia of the whole rod about its middle point is $Ml^2/12$, and $k=l/\sqrt{12}$.

(2) The moment of inertia of a uniform straight thin rod (mass $=M$, length $=l$) about an axis inclined α to the rod and through its end point is $\frac{1}{3}Ml^2\sin^2\alpha$.

(3) Show that the moment of inertia of a uniform thin circular wire (mass $=M$, radius $=r$) about an axis through its centre perpendicular to its plane is Mr^2.



(4) Find the moment of inertia of a uniform thin rectangular plate (mass $=M$, sides $=a$ and b) about an axis through the centre of figure parallel to the side of length b.

The plate may be divided into indefinitely thin strips by lines parallel to the side AB of length a. Then if m_1, m_2, etc., are the masses of the strips, their moments of inertia about the given axis yy' are

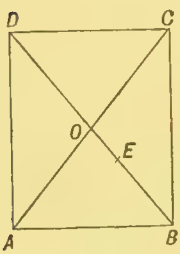
$m_1a^2/12$, $m_2a^2/12$, etc. Hence the moment of inertia of the plate is $(m_1+m_2+\text{etc.})a^2/12=Ma^2/12$.

(5) Show that the moment of inertia of a uniform thin rectangular plate (mass $=M$, sides $=a$ and b) about an axis through its centre of figure perpendicular to its plane is $M(a^2+b^2)/12$. (See 489.)

(6) Find the moment of inertia of a rectangular parallelopiped (mass $=M$) about an axis through the centre of figure perpendicular to one of the faces. (Edges perpendicular to the axis $=a$ and b.)

We may imagine the parallelopiped divided into thin plates by planes perpendicular to the axis. If m_1, m_2, etc., are the masses of these plates, their moments of inertia are (Ex. 5) $m_1(a^2+b^2)/12$, $m_2(a^2+b^2)/12$, etc. Hence the moment of inertia of the parallelopiped is $(m_1+m_2+\text{etc})(a^2+b^2)/12=M(a^2+b^2)/12$.

(7) Find the moment of inertia of a uniform thin right-angled-triangular plate (mass $=M$, sides containing the right angle $=a$ and b) about an axis perpendicular to its plane and through the centre of mass.—Let ABC be the triangular plate and E its centre of mass. Complete the rectangle $ABCD$. E is on the diagonal BD and at a distance from O, the intersection of the diagonals, equal to $\sqrt{a^2+b^2}/6$. Hence the moment of inertia of the triangle about O is (488), if I_E is its moment of inertia about E, $I_E+M(a^2+b^2)/36$. Hence, if I_0 is the moment of inertia of the rectangle $ABCD$ about a normal axis through O,



$$I_0=2[I_E+M(a^2+b^2)/36].$$

But (Ex. 5) $$I_0=2M(a^2+b^2)/12.$$

Hence $$I_E=M(a^2+b^2)/12-M(a^2+b^2)/36$$
$$=M(a^2+b^2)/18.$$

(8) Find the moment of inertia of a uniform thin right-angled-triangular plate (mass $=M$, sides containing the right angle $=a$ and b), about an axis perpendicular to its plane, and through one of the acute angles (say C, fig. of Ex. 7).

The distance of E from C is $\frac{2}{3}\sqrt{b^2+\frac{a^2}{4}}$, if BC is the side of length b. Hence, if I is the required moment of inertia,

$$I=I_E+M\frac{4}{9}\left(b^2+\frac{a^2}{4}\right)$$

$$=M\left(\frac{a^2}{6}+\frac{b^2}{2}\right), \text{ by Ex. 7.}$$

(9) Find the moment of inertia of a uniform thin plate of the shape of an isosceles triangle (mass $=M$) about an axis perpendicular to its plane and through the vertex.

If a is the length of base and h the distance of the vertex from the base, the triangle may be divided into two right-angled triangles whose sides containing the right angle are h and $a/2$, h being the side adjacent to the vertex of the isosceles triangle. Hence

$$I=M\left(\frac{a^2}{24}+\frac{h^2}{2}\right).$$

(10) Find the moment of inertia of a uniform thin plate (mass $=M$) of a regular polygonal shape, about an axis through its centre of figure and normal to its plane.

If there are n sides, each having the length a, and each distant r from the centre of figure, as the polygon may be divided into n isosceles triangles with vertices at the centre of figure, of base a, height r, and mass M/n,

$$I=M\left(\frac{a^2}{24}+\frac{r^2}{2}\right).$$

(11) Find the moment of inertia of a uniform thin circular plate (mass $=M$, radius $=r$) about an axis through its centre and normal to its plane.

As the circle may be considered to be a polygon with an indefinitely great number of indefinitely short sides, each distant r from the centre of figure, we have $I=Mr^2/2$.

(12) Find the moment of inertia of a uniform thin circular plate (mass $=M$, radius $=r$) about (a) a diameter, (b) a tangent.

Ans. (a) $Mr^2/4$; (b) $5Mr^2/4$.

(13) Find the moment of inertia of a uniform circular cylinder (mass $=M$, radius $=r$) (a) about its axis, (b) about a generating line.

Ans. (a) $Mr^2/2$; (b) $3Mr^2/2$.

(14) Find the moment of inertia of a uniform thin spherical shell about a diameter.

If x, y, z be the rectangular coordinates of a point in the shell the centre of the shell being the origin, the moment of inertia about the z axis is $\Sigma m(x^2+y^2)$, m being the mass of the element of the shell at x, y, z. Now by symmetry $\Sigma mx^2 = \Sigma my^2 = \Sigma mz^2$. Hence

$$\Sigma m(x^2+y^2) = \tfrac{2}{3}\Sigma m(x^2+y^2+z^2) = \tfrac{2}{3}\Sigma mr^2 = \tfrac{2}{3}r^2\Sigma m = \tfrac{2}{3}Mr^2,$$

if r is the radius of the sphere, and M its mass.

(15) Find the moment of inertia of a homogeneous sphere about a diameter.

We may imagine the sphere (mass $=M$, radius $=r$) divided into an infinitely large number n of concentric spherical shells of equal thickness r/n. The average of the external and internal radii of the p^{th} shell will be

$$\frac{2p-1}{2}\cdot\frac{r}{n}.$$

Hence if ρ is the density the moment of inertia of this shell will be (Ex. 14),

$$\frac{2}{3}\times 4\pi\rho\left(\frac{(2p-1)r}{2n}\right)^2\cdot\frac{r}{n}\cdot\left(\frac{(2p-1)r}{2n}\right)^2 = \frac{1}{6}\pi\rho\frac{r^5}{n^5}(2p-1)^4.$$

Hence the moment of inertia of the whole sphere will be

$$\begin{aligned} I &= \frac{1}{6}\pi\rho\frac{r^5}{n^5}\{1+3^4+5^4+\text{etc.}+(2n-1)^4\} \\ &= \frac{1}{6}\pi\rho\frac{r^5}{n^5}\left(\frac{16}{5}n^5+\text{terms in lower powers of } n\right) \\ &= \frac{8}{15}\pi\rho r^5, \end{aligned}$$

since we may neglect terms of which $1/n$ is a factor. Hence

$$I = \frac{4}{3}\pi\rho r^3\times\frac{2}{5}r^2 = \frac{2}{5}Mr^2.$$

(16) Find the radius of gyration of a homogeneous sphere (mass $=M$, radius $=r$) about a tangent line.

Ans. $r\sqrt{7/5}$.

491. *Measurement of Moment of Inertia.*—The unit of moment of inertia is that of a particle of unit mass at unit distance from the axis of rotation. In specifying moments of inertia, no mention is usually made of the unit, but they are described as of such and such a value when expressed in such and such units of mass and length.

The dimensions of the unit of moment of inertia are clearly $[M]\ [L]^2$.

492. *Examples.*

(1) Express in oz.-in. units a moment of inertia of 20 ft.-lb. units.
Ans. 46,080.

(2) A moment of inertia has the value 500 when expressed in terms of the centimetre and the gramme. Find its value in terms of the metre and the kilogramme.
Ans. 0·00005.

(3) An author speaks of a rectangular parallelopiped (edges 1, 2, and 12 cm. respectively and density 4 grm. per cub. cm.) as having a moment of inertia equal to 1·207 about an axis through its middle point and perpendicular to the face of greatest area. He is known to have employed the cm. as unit of length, and to have worked where g has the value 980·94 cm.-sec. units. What must have been his unit of mass?

Ans. The unit of mass of the C.G.S. system of gravitational units.

(4) By what number must we multiply the value of a moment of inertia expressed in the unit of the ft.-lb.-sec. absolute system in order to determine its value in terms of the unit of the C.G.S. absolute system?
Ans. 421390·7....

(5) At the end of a thin rod, of length 2 ft. and linear density 1 oz. per in., are particles of masses 1 and 2 lbs. respectively. Express its moment of inertia about an axis perpendicular to the rod through a point distant 3 in. from the particle of smaller mass, in the units of (*a*) the absolute, and (*b*) the gravitational in.-oz.-sec. systems ($g=32$ ft.-sec. units).
Ans. (*a*) 17,352; (*b*) 45·18....

493. *Equations of Motion.*—The moment of inertia being thus a quantity capable of determination, we can apply at once, to cases of motion, the equations of 454.

If the motion be about an axis fixed both in the body and in space, the angular acceleration about it $a = \Sigma FP/\Sigma mr^2$, where ΣFP is the algebraic sum of the moments of the external forces, and Σmr^2 the moment of inertia, about the fixed axis.

If the body be quite free to move, the linear acceleration of the centre of mass is given by the equation $\bar{a} = \Sigma F/\Sigma m$; and the angular acceleration about any axis fixed in the body through the centre of mass, by the equation $a = \Sigma FP/\Sigma mr^2$, where ΣFP is the algebraic sum of the moments of the external forces and Σmr^2 the moment of inertia, about this axis.

These accelerations being determined and the initial velocities being given, the final velocities and the displacement may be found (see 251). The above equations are therefore called the equations of motion of a rigid body. We can apply them only in simple cases of the motion of rigid bodies (227), in cases, viz., in which one line in the body has a fixed direction in space. More complex cases require higher mathematical treatment than the readers of this work are supposed able to apply.

494. In many cases, especially when the forces act only for a very short time, it is convenient to have the equations of motion expressed in terms of the impulses of the acting forces rather than of the forces themselves. Let v, ω and $\bar{v}'$, ω' be the initial and final values of the component linear velocity of the centre of mass in the direction of the impulse, and of the angular velocity of the body about the fixed axis, respectively. Then (117, 225, and 319),

$$\bar{v}' - \bar{v} = \frac{\Sigma Ft}{\Sigma m} = \frac{\Sigma \Phi}{\Sigma m}, \quad \text{and} \quad \omega' - \omega = \frac{\Sigma FPt}{\Sigma mr^2} = \frac{\Sigma \Phi P}{\Sigma mr^2},$$

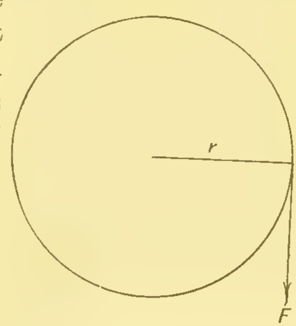
where Φ is the impulse of the force F.

495. The laws of the conservation of linear and of angular momentum, deduced from the equations of motion of extended systems (416 and 429) apply necessarily to rigid systems. The expression of the latter becomes somewhat modified however. In the case of extended systems, it is expressed by the equation $\Sigma m\omega r^2 = \text{constant}$. In the case of a rigid body, either about an axis fixed both in it and in space, or about an axis fixed in it through the centre of mass, it is expressed by the equation $\omega\Sigma mr^2 = \text{constant}$.

496. *Motion about Fixed Axes.*—We shall now discuss some examples of the application of the equations of motion to the determination of the motion of rigid bodies. We take first cases in which the axis of rotation is fixed both in the body and in space.

Examples.

(1) Find the angular acceleration of a uniform circular disc, moveable about an axis through its centre, perpendicular to its plane, under a force applied in the plane of the disc by means of a string fixed at a point of the rim of the disc and wrapped round the rim.—Let the disc have a mass M and a radius r, and let its radius of gyration about the given axis be k. The force F acts tangentially to the disc. If it acts as in the diagram, its moment about the fixed axis is (425) $-Fr$. Hence (493)



$$a = -\frac{Fr}{Mk^2}.$$

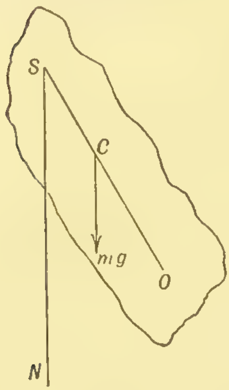
The angular acceleration is therefore constant. Hence, if the initial angular velocity be given, the final angular velocity and the displacement after any time may be determined as in 225.

(2) A rigid rod, 12 ft. long, whose mass may be neglected, has at one end a particle of 10 lbs. mass and at the other a particle of 5 lbs. mass. It is free to rotate in a horizontal plane about an axis

through the centre of mass of the system. Find the force which must be applied to the smaller particle perpendicularly to the rod that unit angular velocity may be produced in the rod in 1 sec.

Ans. 60 poundals.

(3) Find the time of oscillation of a heavy body capable of rotation about a fixed horizontal axis, which does not pass through its centre of mass. [Such a body is called a *compound or physical pendulum.*]—Let the plane of the diagram be a plane through the centre of mass perpendicular to the fixed axis. Let S be the point in which that plane intersects the axis, C the centre of mass, and SN a vertical line.



The external forces are the weight mg (if m is the mass of the body) and the forces by which the axis is fixed. The latter have no moment about the axis. The moment of the weight which (474) acts at the centre of mass is, if we denote CS by h and the angle CSN by θ, $-mgh\sin\theta$. If k is the radius of gyration of the body about an axis parallel to the fixed axis through the centre of mass, the moment of inertia of the body about S is (488) $m(k^2+h^2)$. Hence, if α is the angular acceleration about S,

$$\alpha = -\frac{mgh\sin\theta}{m(k^2+h^2)}.$$

The angular acceleration therefore varies with the displacement from the position in which SC is vertical, and the determination of the displacement produced in any time is therefore difficult.

If however the body move so that θ is always small, we may write θ for $\sin\theta$, in which case

$$\alpha = -\frac{gh}{k^2+h^2}\theta.$$

In a similar way, or by reference to 187,* it may be shown that the

* In 187 we found the tangential acceleration of the bob of a simple pendulum, moving under acceleration g, to be $g\theta$. Now, as the bob moves in a circle, it follows from 135, 130, and 120, that the magnitude of its angular acceleration about the centre is the quotient of its tangential acceleration by its distance from the centre, or in this case $g\theta/l$.

angular acceleration of a simple (or mathematical) pendulum of length l oscillating in a plane is

$$a = -\frac{g}{l}\theta.$$

Hence the motion of the physical pendulum will be the same as that of a simple pendulum whose length

$$l = \frac{k^2 + h^2}{h},$$

and the time of oscillation (187) will therefore be

$$t = 2\pi\sqrt{\frac{k^2 + h^2}{gh}}.$$

A simple pendulum of the length l is usually spoken of as the isochronous simple pendulum.

Produce SC to O, and make SO equal to l. The point O is called the *centre of oscillation*, the point S being called the *centre of suspension* of the pendulum. Then

$$k^2 = h(l - h) = SC \,.\, CO.$$

Let the point O be now made the centre of suspension, *i.e.*, let the given body be made to oscillate about an axis through O, parallel to the original axis; and let OC be produced to a point S' such that OS' is equal to the length of the simple pendulum which is isochronous with the physical pendulum about the new axis. Then it may be shown as before that

$$k^2 = OC \,.\, CS'.$$

Hence the points S and S' coincide, and the centres of suspension and oscillation are convertible.

[Capt. Kater applied this property of the centres of oscillation and suspension of the physical pendulum to the determination of the value of g. He employed a uniform metallic bar, provided with means of suspension at points A and B, known by calculation to be very nearly in the relative positions of the centres of oscillation and suspension, and provided with means of producing slight

changes in the position of the centre of mass of the bar and in its moment of inertia. He then adjusted the instrument so that its time of oscillation was the same whether A or B was the point of suspension. The time of oscillation thus observed was that of a simple pendulum whose length was the distance of A from B. Hence, in the equation $t=2\pi\sqrt{l/g}$, l and t being known, g could be found. The construction of Capt. Kater's pendulum, the mode of its adjustment, and the best methods of observing its time of oscillation, will be found described in books on laboratory practice.]

(4) A uniform rod, 10 ft. long, is suspended from a point $2\frac{1}{2}$ ft. from one end. Find (a) the position of the centre of oscillation, and (b) the time of a small (double) oscillation.

Ans. (a) $1\frac{2}{3}$ ft. from the other end; (b) 2·67... sec.

(5) A uniform cube is free to turn about one edge which is horizontal. Find the length of the edge that the cube may swing to and fro in a second.

Ans. 0·865... ft.

(6) A pendulum consists of a homogeneous spherical bob of radius r, suspended by a string of length l and of negligible mass. Show that the value of the acceleration of a falling body may be calculated from the time t in which the pendulum makes a complete oscillation from the formula

$$g=(4\pi^2/t^2)\{l+r+2r^2/5(l+r)\}.$$

(7) Compare the times of oscillation of a uniform thin circular plate about a horizontal tangent with that about a horizontal axis through the point of contact perpendicular to the plate.

Ans. $\sqrt{5} : \sqrt{6}$.

(8) Determine the axis of suspension of a uniform rectangular lamina for which the time of oscillation under gravity is a minimum.

Ans. Its distance from the centre of mass is equal to the radius of gyration about a parallel axis through that point.

(9) A uniform straight rod AB is freely moveable about its fixed lower end A. The other end B is attached by a fine string to a

fixed point C. The system is slightly displaced, the string being kept tense. Find the time of a small (double) oscillation.

Ans. $t=2\pi\sqrt{\dfrac{2AB\sin\beta}{3g\cos\phi}}$, where ϕ is the inclination of AC to the horizon, and β is the inclination of AC to AB.—AC is the fixed axis of the system. If mg is the weight of AB, its component in a plane perpendicular to AC is $mg\cos\phi$, and the distance of the line of action of this component from AC is, if θ is the inclination of the plane ABC to the vertical plane through AC, $\frac{1}{2}AB\sin\beta\sin\theta$.

(10) A plane lamina is moveable about a fixed point O in its own plane. To determine the line of action of a blow of impulse Φ which will produce no jerk at the fixed point.—Let C be the centre of mass of the lamina. The blow will produce in O two component accelerations, one due to the angular acceleration about C and therefore perpendicular to OC, and the other the translational acceleration common to it and to C and therefore codirectional with the impulse. Since O is fixed and these component accelerations therefore equal and opposite, the line of action of the impulse must be perpendicular to OC. Let $OC=h$, and let d be the distance of Φ's line of action from O, and let k be the radius of gyration of the lamina about C. Then (244), equating the values of the integral angular acceleration about O and C respectively, we have

$$\frac{\Phi d}{m(k^2+h^2)}=\frac{\Phi(d-h)}{mk^2}.$$

Hence
$$d=(k^2+h^2)/h.$$

The point in which Φ's line of action cuts OC is called the *centre of percussion*. By Ex. 3 it coincides with the centre of oscillation.

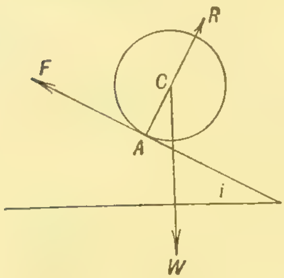
(11) A uniform rod AB is capable of rotation about A. Show that, if a blow (impulse $=\Phi$) be applied perpendicularly to its length at either the point B or a point distant from B by the amount $2AB/3$, the jerk at A is $\Phi/2$, *i.e.*, a blow of impulse $\Phi/2$ must be applied at A to keep that point at rest.

497. *Motion of Free Rigid Bodies.*—The following problems illustrate the application of the equations of motion to bodies which have no point fixed:

Examples.

(1) A uniform circular disc whose plane is vertical rolls (withou sliding) down an inclined plane. Determine its motion.

The disc is acted upon by three forces, its weight W, the normal reaction of the plane R, and the friction F. All are in the plane of the disc. Were there no friction the disc would slide down the plane. Since the friction prevents the sliding, the direction of F is therefore up the plane.



Let the radius of the disc be r, its mass m, and k its radius of gyration about an axis through C perpendicular to its plane, and let i be the inclination of the plane. Then, if $\bar{a}$ is the linear acceleration of the centre of mass down the plane,

$$\bar{a} = g \sin i - F/m.$$

The linear acceleration in a direction normal to the plane is zero. Hence the above is the resultant linear acceleration. The resultant angular acceleration about the centre of mass is

$$a = -\frac{Fr}{mk^2}.$$

Now (254, Ex. 9) $\qquad \bar{a} + ar = 0.$

Hence $\qquad \bar{a} = \frac{Fr^2}{mk^2}, \quad \text{and} \quad F = \frac{m\bar{a}k^2}{r^2}.$

Substituting in the first equation this value of F, we find

$$\bar{a} = \frac{gr^2 \sin i}{k^2 + r^2}.$$

Hence $\qquad a = -\frac{gr \sin i}{k^2 + r^2}.$

Both linear and angular accelerations are therefore constant. Hence the displacements and velocities after any given time may readily be determined.

(2) Find the time a rigid cylinder will take to roll from rest down a plane 20 ft. long and inclined 30° to the horizon, the axis of the cylinder being horizontal.

Ans. 1·93... sec.

(3) A uniform circular disc whose plane is vertical moves in contact with a smooth inclined plane. From a point in the same vertical plane as the disc and at a distance from the inclined plane equal to the diameter of the disc, a string is carried parallel to the inclined plane and is wrapped round the edge of the disc and its end is fixed in the circumference. Find (a) the tension in the string, (b) the linear acceleration of the centre of the disc, and (c) its angular acceleration.

Ans. (a) $mgk^2 \sin i/(k^2+r^2)$, (b) $gr^2 \sin i/(k^2+r^2)$, (c) $gr \sin i/(k^2+r^2)$, where m is the mass of the disc, r its radius, k its radius of gyration about an axis through its centre perpendicular to its plane, and i the inclination of the inclined plane.

(4) A flexible and inextensible ribbon is coiled on the circumference of a uniform circular disc (radius $=r$) and has its free end attached at a fixed point. A part of the ribbon is unrolled and vertical, and the disc is allowed to fall from rest by its own weight. Find (a) the motion of the disc before the ribbon becomes wholly unrolled, and (b) the time in which the centre of the disc will descend $g/3$ feet from rest.

Ans. (a) Acceleration of centre of mass $=2g/3$; angular acceleration $=2g/(3r)$; (b) 1 sec.

(5) A homogeneous hemisphere performs small oscillations on a perfectly rough horizontal plane. Find the periodic time.

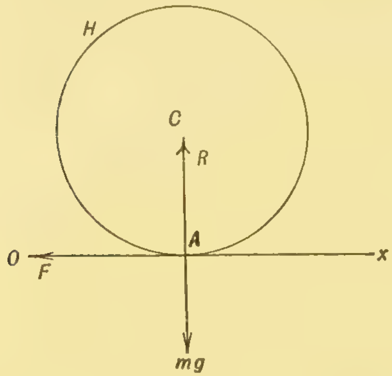
Ans. If r is the radius, c the distance of the centre of mass from the centre of the hemisphere, and k the radius of gyration about an axis parallel to the instantaneous axis through the centre of mass, the time of a small oscillation is

$$2\pi\sqrt{\frac{k^2+(r-c)^2}{cg}}.$$

(6) A uniform circular hoop moving in a vertical plane in contact with a rough horizontal surface has at a given instant an angular velocity opposite in direction to that which would enable it to roll in the direction of its translation at that instant. Determine its subsequent motion.

Let H be the hoop, Ox the intersection of the given vertical and horizontal planes. The forces acting on the hoop are its weight

and the components of the reaction of the plane, viz., its normal



component R and the friction F. If we suppose the translation of the hoop to be in the direction Ox, the friction F which acts at A will have the direction AO. R and the weight mg act also through A, the one vertically upwards, the other vertically downwards. If m is the mass of the hoop, r its radius, k its radius of gyration about an axis through its centre C perpendicular to its plane, the acceleration of the centre of mass parallel to Ox

$$\bar{a} = -F/m,$$

and the angular acceleration about an axis through C perpendicular to the plane of the hoop

$$\alpha = -Fr/(mk^2).$$

Since the hoop remains in contact with Ox there is no acceleration perpendicular to Ox; and therefore

$$R - mg = 0.$$

Hence, if μ is the coefficient of kinetic friction,

$$F = \mu mg.$$

Hence $$\bar{a} = -\mu g$$

and $$\alpha = -\mu gr/k^2.$$

The linear and angular accelerations are therefore constant, and hence the initial linear and angular velocities, and the initial position of the hoop being given, its position and velocities after any time may be determined.

If at any instant there be no slipping we have (254, Ex. 9), if $\bar{v}'$ and ω' are the linear velocity of C and the angular velocity about C respectively at that instant, $\bar{v}' + \omega' r = 0$. If therefore it be required to determine the time, t, after which slipping ceases, we have,

denoting by $\bar{v}$ and ω the initial values of the linear and angular velocities respectively,

$$v' = \bar{v} - \mu g t,$$

$$\omega' = \omega - \mu g r t / k^2,$$

$$\bar{v}' + \omega' r = 0.$$

Eliminating $\bar{v}'$ and ω' from these equations we find

$$t = \frac{k^2(\bar{v} + \omega r)}{\mu g (k^2 + r^2)}.$$

At the instant at which $\bar{v}' + \omega' r$ becomes zero, there is no tendency to slip, and μ becomes zero. Hence $\bar{a} = 0$ and $a = 0$. Hence after the time t the linear and angular velocities are constant, and their values are

$$\bar{v}' = \bar{v} - \frac{k^2(\bar{v} + \omega r)}{k^2 + r^2} = \frac{r(r\bar{v} - k^2\omega)}{k^2 + r^2},$$

$$\omega' = \omega - \frac{r(\bar{v} + \omega r)}{k^2 + r^2} = \frac{k^2\omega - r\bar{v}}{k^2 + r^2}.$$

If $r\bar{v} - k^2\omega$ is negative, $\bar{v}'$ is negative. Hence if ω is positive and greater than $r\bar{v}/k^2$, the translation of the hoop will, after the above time t, be in the opposite direction to the initial translation.

The above results apply also of course to a ball spinning about a horizontal axis perpendicular to the direction of its translation.

The reader who in his youth has played with a hoop, or in more advanced years has amused himself with a napkin ring or a billiard ball, will recognise in the above results the mathematical expression of a familiar experience.

(7) A uniform circular cylinder (radius $= r$, radius of gyration about axis $= k$), rotating about its axis with angular velocity, ω, is placed with its axis horizontal on a rough inclined plane (coefficient of friction $= \mu$, inclination (i) to horizon $= \tan^{-1}\mu$), the direction of the rotation being that which it would have if the cylinder were rolling without sliding up the plane. Show that the axis of the cylinder will be stationary for a time $k^2\omega/(\mu r g \cos i)$, at the end of which the angular velocity will be zero.

(8) A uniform beam is supported horizontally on two props. Where must one of them be placed that, when the other is removed

the instantaneous force exerted on the former may be equal to half the weight of the beam?

Let h be the required distance of the permanent prop from the middle point of the beam, k the radius of gyration of the beam about a normal axis through its middle point, m the mass of the beam, α its angular acceleration, and R the force exerted on the permanent prop immediately after the removal of the other. Then (496, Ex. 9, and 244),

$$\alpha=\frac{mgh}{m(k^2+h^2)}=\frac{Rh}{mk^2}=\frac{\frac{1}{2}mgh}{mk^2}.$$

Hence $h=k$.

(9) A uniform square is supported in a vertical plane with one diagonal horizontal by two pegs, one at each extremity of the diagonal. Show that the initial force on one peg, when the other is suddenly removed, is equal to one fourth of the weight of the square.

(10) A uniform horizontal bar, suspended from any two points in its length by two parallel cords, is at rest. If one of the cords be cut, find the initial tension in the other.

Ans. If l is the length of the bar, W its weight, and d the distance from its centre of mass of the point of attachment of the uncut cord, the tension is $Wl^2/(l^2+12d^2)$.

(11) A uniform beam (weight $=W$) rests with one end against a smooth vertical wall, and the other on a smooth horizontal plane, its inclination to the horizon being i. It is prevented from falling by a string attached to its lower end and to the wall. Find the instantaneous force between the upper end and the wall when the string is cut.

Ans. $\frac{1}{2}W\cot i$.

(12) A sphere is laid upon a rough inclined plane (inclination $=i$). Show that it will not slide, if the coefficient of friction is as great as, or greater than, (2/7) $\tan i$.

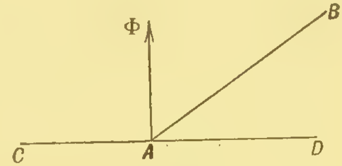
(13) A sphere (radius $=r$) whose centre of mass is not at its centre of figure, is placed on a rough table (coefficient of friction $=\mu$); find whether it will begin to slide or to roll.

Ans. If the initial distance of the centre of mass from a vertical line through the centre of figure is greater than $\mu k^2/r$, k being the

radius of gyration of the sphere about the tangent line through its point of contact with the table and perpendicular to the plane containing its centres of figure and of mass, it will begin to slide; if less, to roll.

(14) One end of a rigid rod which is moving in any way in a vertical plane impinges upon a fixed smooth horizontal plane (a) without, or (b) with, recoil. It is required to determine the initial motion of the beam after impact.

Let AB be the rod, CD the intersection of the plane in which the rod is moving, with the horizontal plane. Let θ be the inclination of the rod to the plane at the instant of impact, m the mass of the rod, a the distance of its centre of mass from A, k its radius of gyration about an axis through its centre of mass perpendicular to the plane of its motion.



Let Φ be the impulse experienced by the rod at A on impact. Its direction is vertically upwards. Let $\bar{v}$, $\bar{v}'$, be the components vertically upward of the velocities of the centre of mass before and after impact respectively. Let ω and ω' be the angular velocities about the centre of mass before and after impact respectively (the positive direction of rotation being counter-clockwise). Then the integral linear and angular acclerations are $\bar{v}' - \bar{v}$ and $\omega' - \omega$ respectively. Hence

$$\bar{v}' - \bar{v} = \Phi / m$$

and
$$\omega' - \omega = -\Phi a \cos\theta / (mk^2).$$

(a) If there is no recoil, A remains in contact with CD after the impact. Hence the sum of its component velocities after impact must be zero. These are $\bar{v}$ upwards and $\omega' a \cos\theta$ downwards. Hence

$$\bar{v}' - \omega' a \cos\theta = 0.$$

These three equations are sufficient to determine $\bar{v}'$, ω', and Φ in terms of $\bar{v}$, ω, and θ.

(b) If there is recoil, let the coefficient of restitution be e. Then if Φ' be the impulse on impact, and Φ the value the impulse would have were there no recoil, we have (379)

$$\Phi' = (1 + e)\Phi.$$

Φ' is therefore known in terms of $\bar{v}$, ω, θ, and e. If its value be substituted for Φ in the first two equations, $\bar{v}'$ and ω' may then be determined for this case also.

(15) A beam which is moving without rotation in a horizontal plane impinges without recoil on a fixed rod at right angles to the plane. Find (a) the impulse of the reaction of the rod, and (b) the angular velocity of the beam immediately after impact.

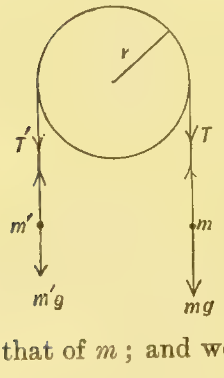
Ans. (a) $mk^2u \sin a/(c^2+k^2)$; (b) $cu \sin a/(c^2+k^2)$; where m = mass of beam, k = radius of gyration about normal axis through centre of mass, u = velocity of centre of mass before impact, a = inclination of direction of u to the beam, c = distance of centre of mass of beam from fixed rod at instant of impact.

498. *Motion of Systems of Rigid Bodies.*—If the motion is to be determined of several bodies which act upon one another, the equations of motion must be applied to each of them. The following cases will serve as illustrations:

Examples.

(1) Two particles of masses m and m' are connected by an inextensible string which hangs over a pulley moveable about a fixed horizontal axis. The axle of the pulley is smooth, its rim so rough that the string does not slip. Find the acceleration of the particles. (*Atwood's Machine.* See 382, Ex. 1.)

Let T and T' be the tensions in the portions of the string attached to m and m' respectively. The moments of T' and T about the axis of the pulley are, if r is the radius of the pulley, $T'r$ and $-Tr$ respectively. Hence, if a is the angular acceleration of the pulley,



$$a = \frac{T'r - Tr}{Mk^2},$$

where M is the mass of the pulley and k its radius of gyration about its axis. As the string is inextensible, the acceleration a of m' is the same as that of m; and we have as in 382, Ex. (1),

$$a = (m'g - T')/m' = (T - mg)/m.$$

We have also, since the string does not slip,

$$a = \alpha r.$$

Hence we have four equations involving four unknown quantities, a, α, T, T'. Eliminating α, T, and T', we find

$$a = \frac{m' - m}{m' + m + Mk^2/r^2} \cdot g\,;$$

which differs from the result of 382, Ex. 1, in which the string hangs over a smooth peg by the introduction of the quantity Mk^2/r^2.

(2) A uniform cylinder weighing 100 lbs. turns without friction on its axis which is horizontal. Motion is communicated by a body of 10 lbs. mass, attached to an inextensible string without weight, which is coiled round the surface of the cylinder. Find the distance through which the body will descend from rest in 10 sec.

Ans. $10g$.

(3) To the string coiled round the wheel of the simple machine called the Wheel and Axle (254, Ex. 5) a mass of 10 lbs. is attached; to the string around the axle a mass of 100 lbs. Given that the radii are 3 ft. and 3 in. respectively, that the moment of inertia about the axis expressed in terms of the pound and foot is 2400, and that the machine is frictionless, find the number of revolutions made in 1 minute from rest, taking g to be 32 ft.-sec. units.

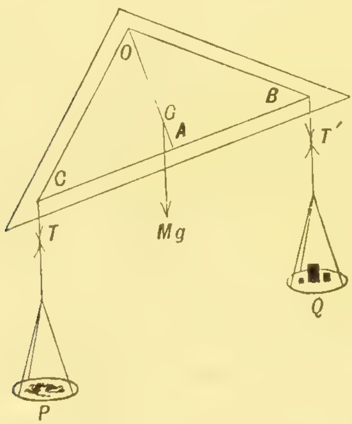
Ans. $60/\pi$.

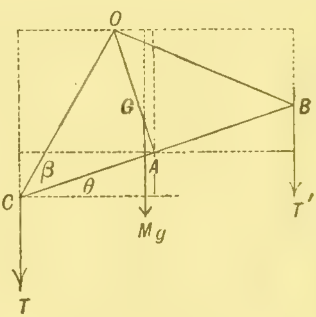
(4) Find the time of a small oscillation of a Balance with nearly equal masses in its pans. [The *Balance* consists of a practically rigid body called the beam, moveable about a horizontal axis fixed in it, and symmetrical about a plane through this axis and the centre of mass of the beam. It carries pans or scales to contain, one the body to be weighed, and the other standards of mass. The pans are moveable about axes fixed in the beam, which are parallel to the axis of the beam, and are equidistant both from it and from the centre of mass. A plane through the centre of mass of the beam, perpendicular to the three axes, intersects them in three points, which are called the points of suspension of the beam and pans respectively. The distances of the points of suspension of the pans from that of the beam are called the arms of the balance. The

centre of mass of the beam is usually below its fixed axis, the beam being provided with an adjustment by which the position of that point may be varied. The line joining the points of suspension of the pans passes in general below the axis of the beam. In some instruments it is made to pass as nearly as possible through the axis, in others a little above it, when the pans are unloaded.]

Let OBC be the beam, O being its point of suspension, G its centre of mass, and B and C the points of suspension of the pans P and Q. Let A be the point in which OG produced cuts CB. It is obvious that OA is at right angles to BC. In the diagram the angle OCA is for clearness made large. It is usually small.

The beam is acted upon by three forces, its weight and the resultants T, T' of the tensions in the strings or rods supporting P and Q respectively. The motion of the beam is usually slow and through small angles. Hence though P and Q will oscillate about C and B, we may for an approximate result assume T and T' to be





vertical. If, then, M be the mass of the beam and k its radius of gyration about O, m and m' the masses of the pans P and Q respectively with their contents, β the angle OCA, and θ the inclination of BC to the horizon at any instant, we have for the angular acceleration (α) of the beam

$$\alpha = \frac{T(AC\cos\theta - OA\sin\theta) - T'(AB\cos\theta + OA\sin\theta) - Mg.OG.\sin\theta}{Mk^2}.$$

For the reader will have no difficulty in proving by the aid of the

second figure (in which the dotted lines are all either horizontal or vertical) that the numerator of this expression for a is the algebraic sum of the moments of the forces about O.

The pans P and Q are acted upon by two forces each, viz., their weights and the resultants T and T' of the tensions in the strings or rods supporting them. Hence the vertical linear acceleration of P is $(mg - T)/m$; and it is equal to the vertical component of the linear acceleration of C, which is $a \,.\, OC \,.\, \cos(\theta + \beta)$. Hence

$$mg - T = maOC\cos(\theta+\beta).$$

Similarly $$T' - m'g = m'aOC\cos(\theta-\beta).$$

Now $$OC\cos(\theta\pm\beta) = OC(\cos\theta\cos\beta \mp \sin\theta\sin\beta)$$
$$= AC\cos\theta \mp OA\sin\theta.$$

Hence

$$aMk^2 = mg(AC\cos\theta - OA\sin\theta) - m'g(AC\cos\theta + OA\sin\theta)$$
$$- MgOG\sin\theta - maOC^2\cos^2(\theta+\beta) - m'aOC^2\cos^2(\theta-\beta).$$

For small values of θ therefore

$$aMk^2 = mg(AC - OA \,.\, \theta) - m'g(AC + OA \,.\, \theta) - Mg \,.\, OG \,.\, \theta$$
$$-(m+m')aOC^2\cos^2\beta.$$

Hence, noting that $AC = OC\cos\beta$,

$$a = \frac{m(AC - OA\,.\,\theta) - m'(AC + OA\,.\,\theta) - M\,.\,OG\,.\,\theta}{Mk^2 + (m+m')AC^2}\,g.$$

If the masses of P and Q are the same, viz. m, we have

$$a = -\frac{2m\,.\,OA + M\,.\,OG}{Mk^2 + 2m\,.\,AC^2}\,g\,\theta.$$

Hence (496, Ex. 3) the time of a small oscillation is

$$t = 2\pi\sqrt{\frac{Mk^2 + 2m\,.\,AC^2}{(2m\,.\,OA + M\,.\,OG)g}}.$$

If B, O, and C are in the same straight line, $AC = OC$ and $OA = 0$.

Hence $$t = 2\pi\sqrt{\frac{Mk^2 + 2m\,.\,OC^2}{M\,.\,OG\,.\,g}}.$$

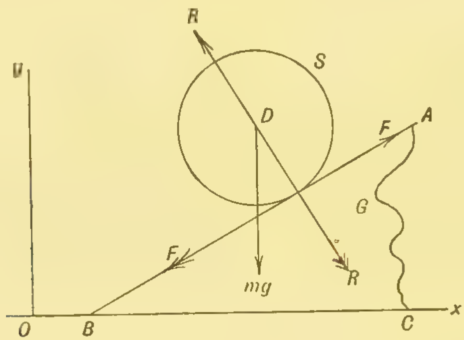
Quickness of motion is a desirable characteristic of a balance, and it should therefore be so constructed that t may be as small as possible. Hence for this purpose the radius of gyration of the beam, and the distance between the points of suspension of

the pans, should be as small as possible, and the distance from the axis both of the centre of mass and of the line joining the points of suspension of the pans should be as great as possible.

For conditions of sensitiveness, see 507, Ex. 11.

(5) A uniform sphere S rolls without sliding down AB a line of greatest slope of the inclined plane surface of a wedge AC, which lies upon a smooth horizontal table Ox, the centres of mass of the sphere and wedge and the given line AB being in the same vertical plane. It is required to determine the motion of the sphere and wedge.

Let the vertical plane containing the line AB and the centres of mass of sphere and wedge, D and G, intersect the table in the line Ox; and let Oy be drawn in this plane, and perpendicular to Ox. Let m be the mass of the sphere, r its radius, k its radius of gyration about any axis through its centre of mass, m' the mass of the



wedge, R the normal component, and F the frictional component of the reaction between the sphere and wedge, and ϕ the angle ABC of the wedge. As it is the frictional component of the reaction of the wedge on the sphere which causes the sphere to roll, it must be directed up the plane. Hence the same component of the reaction of the sphere on the wedge is directed down the plane. (The forces acting on the wedge are indicated by double arrow-heads.)

The equations determining the motion of the sphere are as follows. If $\bar{a}_x$ is the linear acceleration of the centre D of the sphere in the direction Ox,

$$\bar{a}_x = (-R\sin\phi + F\cos\phi)/m.$$

If $\bar{a}_y$ is its linear acceleration in the direction Oy,

$$\bar{a}_y=(R\cos\phi+F\sin\phi-mg)/m.$$

If a is the angular acceleration of the sphere about an axis through D perpendicular to the plane of the diagram

$$a=Fr/(mk^2).$$

The wedge obviously moves so that BC remains in contact with the table. Hence the weight of the wedge and the normal reaction of the smooth table do not affect its motion, and the acceleration $\bar{a}'_x$ of the centre of mass of the wedge is therefore

$$\bar{a}'_x=(R\sin\phi-F\cos\phi)/m'.$$

In the above four equations we have six unknown quantities, R, F, a, $\bar{a}_x$, $\bar{a}_y$, $\bar{a}'_x$. But two more equations may be obtained by a statement of the kinematic conditions of the problem. First, since S rolls down the inclined plane, the change produced in any time in D's vertical distance from any point B in the wedge divided by the change in its horizontal distance from the same point is equal to $\tan\phi$. Hence at every instant the ratio of D's vertical velocity relative to B to its horizontal velocity relative to B has this value; and hence also the ratio of the vertical acceleration of D to its horizontal acceleration, relative to B, has the value $\tan\phi$. Now, the vertical accelerations of D relative to B and to O are the same, for B has no vertical acceleration relative to O. And the horizontal acceleration of D relative to O is equal to that of B relative to O, together with that of D relative to B; and therefore the horizontal acceleration of D relative to B is equal to that of D relative to O, *minus* that of B relative to O. Hence the first kinematic condition may be expressed by the equation

$$\bar{a}_y/(\bar{a}_x-\bar{a}'_x)=\tan\phi.$$

Secondly, since there is no sliding of the sphere on the inclined plane, the linear velocity in any direction of that point of the sphere which is in contact with the wedge must at every instant be equal to the velocity of the wedge in the same direction. Now, this point of the sphere has in the direction Ox two component linear velocities, one v_x the horizontal velocity of D, and another

$r\omega\cos\phi$ due to the angular velocity ω of the sphere about D. Hence if v_x' is the horizontal velocity of the wedge,

$$v_x' = v_x + r\omega\cos\phi.$$

As this holds for every instant, it follows that

$$\bar{a}'_x = \bar{a}_x + r\alpha\cos\phi.$$

The six equations thus obtained are sufficient to determine the values of all the unknown quantities they contain.

(6) A uniform cylinder (mass $= m$) rolls, without sliding, directly down a rough inclined plane (inclination $= i$), while a string coils round it which unwinds from an equal parallel cylinder, revolving about its axis which is fixed, the position of the latter cylinder being such that the string is parallel to the plane. Find (a) the linear acceleration of the cylinder; (b) the stress in the string; and (c) the friction of the inclined plane.

Ans. (a) $\frac{2}{7}g\sin i$; (b) $\frac{2}{7}mg\sin i$; (c) $\frac{3}{7}mg\sin i$.

(7) Two rigid discs, of radii r and r' and masses m and m', can rotate in their common plane about axes through their central points. They nearly touch each other, and each has a small projecting tooth. The one whose mass is m' is at rest. The other has an angular velocity ω_0. Find their angular velocities after the impact of the teeth, (a) if there is no recoil, (b) if the coefficient of restitution is e.

(a) Let ω be the common angular velocity after impact and Φ the magnitude of the impulse. Then (494 and 490, Ex. 11),

$$\omega - \omega_0 = -\Phi r/(\tfrac{1}{2}mr^2),$$

and
$$\omega = \Phi r'/(\tfrac{1}{2}m'r'^2).$$

Hence
$$\omega = \omega_0 mr/(mr + m'r').$$

(b) Let ω and ω' be the magnitudes of the respective angular velocities after impact, and ϕ the impulse. Then

$$\omega - \omega_0 = -\phi r/(\tfrac{1}{2}mr^2),$$

and
$$\omega' = \phi r'/(\tfrac{1}{2}m'r'^2).$$

Now (379)
$$\phi = \Phi(1 + e),$$

and from the equations of (a)

$$\Phi = \frac{mm'rr'\omega_0}{2(mr + m'r')}.$$

Hence $$\omega = \omega_0 \left(1 - \frac{m'r'(1+e)}{mr + m'r'}\right).$$

and $$\omega' = \omega_0 \cdot \frac{mr(1+e)}{mr + m'r'}.$$

(8) A particle, of mass m, dropped from a height l, strikes the end of a horizontal uniform beam, of mass $3m$ and length $2l$, which can move freely about its centre of mass. Find the angular velocity of the beam immediately after impact, (a) if there is no recoil, (b) if the coefficient of restitution is 0·5.

Ans. (a) $\sqrt{g/2l}$; (b) $\frac{3}{2}\sqrt{g/2l}$.

(9) A circular cylinder (mass $= m$, radius $= r$, radius of gyration about its axis $= k$) is revolving with angular velocity ω about its axis which is horizontal, when it suddenly begins to lift a particle (mass $= m'$) by means of an inextensible string wound round the cylinder. Find (a) the angular velocity of the cylinder immediately after the particle begins to move, and (b) the impulse of the stress transmitted by the string.

Ans. (a) $\dfrac{\omega m k^2}{m'r^2 + mk^2}$; (b) $\dfrac{\omega m m' r k^2}{m'r^2 + mk^2}$.

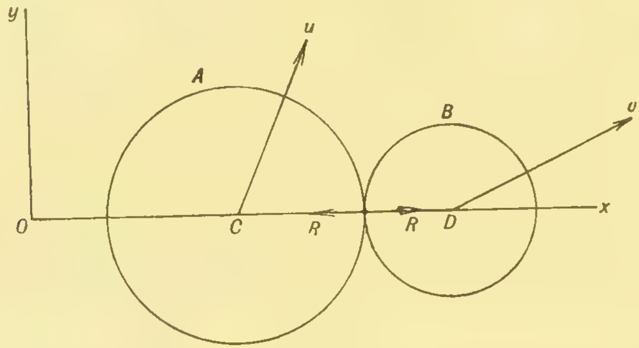
(10) Two uniform weightless spheres, either smooth or rough, and moving without rotation, undergo impact. At the instant of impact, their centres are moving in the direction of the line joining them. Given their velocities before impact, find their velocities after impact.

Since the spheres are either smooth or moving without rotation, the direction of the stress during impact is normal to the surfaces of the spheres. It therefore produces no angular acceleration. The equations of motion are therefore the same as in the case of two particles moving in the same straight line which so impinge, that the stress during impact is in the line of motion. Hence the results of 380, Ex. 1, apply also to this problem.

When two spheres impinge whose centres at the instant of impact are moving in the direction of the line joining them, the

impact is said to be *direct*. From the above it is obvious that Exs. 1–8 of 380 may all be transformed, by changing a few words, into examples of the direct impact of smooth or of non-rotating spheres.

(11) Two uniform smooth weightless spheres, moving without rotation so that their centres remain in a given plane, impinge with or without recoil. Given their velocities immediately before impact to find them immediately after.—Let A, B be sections of



the spheres by the given plane, C, D their centres of figure and therefore of mass. Let u, v, inclined $\phi°$ and $\chi°$ to CD respectively, be their velocities before impact, u', v', inclined $\phi'°$, $\chi'°$ respectively to CD, their velocities after impact.

During impact a stress of impulse R acts between the spheres in the line joining their centres. It therefore produces no angular acceleration in either. As there are no external forces acting on the spheres, the linear momentum (416) in the direction of the line of impact (CD) is the same before and after impact. Hence, if M and m, are the masses of the respective spheres,

$$Mu\cos\phi + mv\cos\chi = Mu'\cos\phi' + mv'\cos\chi'.$$

As no forces act on either sphere in the direction perpendicular to CD

$$u\sin\phi = u'\sin\phi',$$

and

$$v\sin\chi = v'\sin\chi'.$$

If there is no recoil the component velocities in the direction of CD are simply equalized by the impact. Hence

$$u'\cos\phi' = v'\cos\chi'.$$

These four equations are sufficient to determine u', v', ϕ', χ' in terms of u, v, ϕ, and χ.

If there is recoil, and if e is the coefficient of restitution, we have (379)

$$v' \cos \chi' - u' \cos \phi' = e(u \cos \phi - v \cos \chi);$$

and this equation, with the first three of those obtained above, are sufficient to determine the four unknown quantities.

The impact of two spheres, whose centres at the instant of impact are not moving in the same straight line, is said to be *oblique*.

(12) A smooth ball A, weighing 20 grm., strikes another ball B which is at rest, the direction of A's motion being inclined 30° to the line joining the centres of A and B at the instant of impact, and glances off in a direction perpendicular to that of its motion before impact. Find the mass of B, the coefficient of restitution being 0·4.

Ans. 400 grm.

(13) A billiard ball A (mass $= m$) impinges upon another B (mass $= m'$) which is at rest, the direction of A's motion before impact being inclined 45° to the line joining the centres at the instant of impact. Find the direction of A's motion after impact, assuming the coefficient of restitution equal to unity.

Ans. Inclination to the line joining the centres $= \tan^{-1} \dfrac{m+m'}{m-m'}$.

(14) Two billiard balls A and B are lying in contact on a table. Find the direction in which B must be struck by a third ball C so as to be driven off in a direction inclined at a given angle θ to the line joining the centres of A and B, all three balls being smooth and of equal volume and mass. Show that the result is the same whatever be the value of the coefficient of restitution.

Ans. The line joining the centres of C and B at the instant of impact must be inclined to the line joining the centres of A and B at the angle $\tan^{-1}(\frac{1}{2} \tan \theta)$.

(15) Two straight rods ACB and CD, whose thickness and density are equal, and whose coefficient of restitution is unity, lie on a smooth horizontal plane at right angles to each other, the end C

of the latter being in contact with the former. Determine the point at which ACB may be struck without consequent rotation.

Ans. If $AC=a$, $CB=b$, and $CD=c$, and if $a>b$, the required point is in AC, and its distance from C is $\dfrac{a^2-b^2}{2(a+b+c)}$.

(16) A ball (mass $=m$, radius $=r$, radius of gyration about its centre $=k$) sliding without rotation along a smooth horizontal plane, with velocity u, strikes against a perfectly rough vertical plane, its direction of motion before impact being inclined at the angle θ to the vertical plane. Show (1) that if there is no recoil the impulse, during impact, of the frictional component of the reaction of the vertical plane is

$$\frac{mk^2u\cos\theta}{r^2+k^2}.$$

and (2) that if the coefficient of restitution is e, the ball's direction of motion after impact is inclined to the vertical plane at the angle $\tan^{-1}\left(\dfrac{e(r^2+k^2)}{r^2-ek^2}\tan\theta\right)$.

499. *The Law of Energy.*—The general law of energy, deduced (436) from the equations of motion for extended systems, including the law of the conservation of energy (434), applies of course to those extended systems which are rigid. Its application to the solution of problems is simplified in the case of rigid bodies for two reasons. First, as the particles of the system are at invariable distances from one another, the internal forces do no work in any displacement, and therefore the external forces only appear in the equation. Secondly, the expression for the kinetic energy relative to the centre of mass is very simple. If ω is the angular velocity about an axis fixed in the body, and r the distance of a particle from the axis, ωr is its linear velocity relative to points in that axis. Hence the kinetic energy of the system relative to points in the given axis is

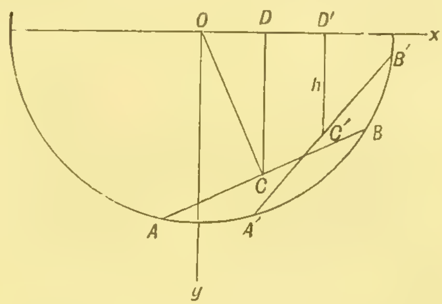
$$\tfrac{1}{2}\Sigma m\omega^2r^2=\frac{\omega^2}{2}\Sigma mr^2=\tfrac{1}{2}Mk^2\omega^2,$$

if M and k are the mass of the body and its radius of gyration about the given axis respectively. If the given axis pass through the centre of mass, the above is the expression for the kinetic energy relative to that point.

We shall illustrate the application of the law of energy to the solution of kinetic problems by a few examples:

Examples.

(1) A uniform rod moves in a vertical plane within a fixed smooth hemisphere. To determine its angular velocity in any position, its initial position being one of instantaneous rest.—Let ABB' be the hemisphere, O its centre, Ox and Oy horizontal and vertical lines respectively in the plane of the rod's motion. Let $A'B'$ be the initial position of the rod, h being the distance $(C'D')$ of its centre of mass (C') from Ox. Let AB be its position at the instant under consideration, OD and DC, or $\bar{x}$ and $\bar{y}$, being the co-ordinates of its centre of mass.



The component velocities of C will be $\dot{\bar{x}}$ and $\dot{\bar{y}}$. Hence if ω is the angular velocity of the rod about C, m its mass, and k its radius of gyration about a normal axis through C, the kinetic energy of the rod in the position AB and therefore the increase of kinetic energy during the motion from the position $A'B'$ to the position AB is (442) $\frac{1}{2}m(\dot{\bar{x}}^2+\dot{\bar{y}}^2+k^2\omega^2)$. The external forces acting on the rod are the reactions of the smooth sphere and the weight of the rod. As the ends of the rod move in all positions in directions perpendicular to the reactions exerted on them, no work is done by or against these reactions. Work has been done by the weight of the rod, and, as the centre of mass has fallen through the distance $\bar{y}-h$ vertically, the potential energy has diminished by the amount $mg(\bar{y}-h)$. Hence, by the law of conservation of energy,

$$\tfrac{1}{2}m(\dot{\bar{x}}^2+\dot{\bar{y}}^2+k^2\omega^2)-mg(\bar{y}-h)=0.$$

Now the instantaneous centre of the motion (233) of AB is O. Hence

the linear velocity of C is perpendicular to OC and has the magnitude $OC\,.\,\omega$, or, if c be the distance of the rod from the centre, $c\omega$. Hence its components in the direction Ox, Oy are, if the angle COy be written θ,

$$\dot{\bar{x}} = -c\omega\cos\theta,$$

$$\dot{\bar{y}} = c\omega\sin\theta.$$

Also $$\bar{y} = c\cos\theta,$$

and $$h = c\cos\phi,$$

if ϕ is the initial value of θ. Hence, substituting in the above equation these values of $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\bar{y}$, and h, we obtain

$$(c^2+k^2)\,\omega^2 = 2cg(\cos\theta - \cos\phi).$$

If $2a$ is the length of the rod, $k^2 = a^2/3$. Hence

$$\omega^2 = \frac{6cg}{3c^2+a^2}(\cos\theta - \cos\phi).$$

(2) Two equal spheres starting at the same instant without initial velocity move down two equally inclined planes, one of which is smooth, the other perfectly rough. Find the ratio of the kinetic energy of the former sphere to that of the latter at the end of any time. (See 490, Ex. 14.)

Ans. 7/5.

(3) A solid cylinder is freely moveable about its axis which is fixed horizontally, and masses m, m' are hung at the ends of a string wound round it and attached to it at some point so as to prevent slipping. After m' has descended from rest for t seconds it is suddenly cut off and the system comes to rest in t seconds more. Find the mass of the cylinder.

Ans. $4m^2/(m'-2m)$.

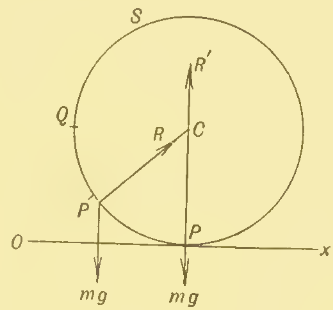
(4) A uniform rod AB (length $=2a$, mass $=m$, radius of gyration about a normal axis through the centre of mass $=k$) is capable of moving freely about a hinge at a point A in a smooth horizontal table. The other end B rests upon the smooth upper surface of a wedge (angle $=i$, mass $=m'$) which lies upon the table, the vertical plane through AB being perpendicular to the edge of the wedge and passing through its centre of mass. Find the angular velocity

of the rod when inclined to the table at an angle θ, given that its value was zero when the inclination of the rod was β.

Ans. $\left(\dfrac{2mag\sin^2 i(\sin\beta-\sin\theta)}{m(a^2+k^2)\sin^2 i+4m'a^2\cos^2(i-\theta)}\right)^{\frac{1}{2}}$.

(5) A thin spherical shell rests upon a smooth horizontal plane and a particle of the same mass as the shell is placed at the lowest point of its internal surface, which is smooth. With what horizontal velocity must the shell be projected in order that the particle may ascend to the height of the centre of the shell.

Let C be the centre of the shell, S its section by a vertical plane through C, and Ox the direction of projection. The particle is to move from P up to Q. At any position P' it is acted upon by two forces, its weight mg and the reaction R of the shell. The forces acting on the shell are its weight mg, the reaction R' of the horizontal plane, and a force equal and opposite to R. All pass through its centre of mass, and therefore (495) the angular acceleration of the shell being initially zero continues to be zero. When the particle has risen to the height of the centre it is to be instantaneously at rest relatively to the shell. Hence it will have the same velocity as the shell. Let v' be the velocity of shell and particle at that instant, v the initial velocity of the shell, m its mass, and a its radius. Then the initial kinetic energy of the system is $\frac{1}{2}mv^2$ and the final kinetic energy is mv'^2. The increase of the potential energy of the particle is mga. The potential energy of the shell undergoes no change. Hence, by the law of the conservation of energy,



$$mv'^2-\tfrac{1}{2}mv^2+mga=0.$$

Now the external forces acting on the system of shell and particle are all vertical. Hence, by the law of the conservation of linear momentum (416), the horizontal momentum is constant and we have thus

$$mv=2mv'.$$

Eliminating v' from these equations we find

$$v=2\sqrt{ga}.$$

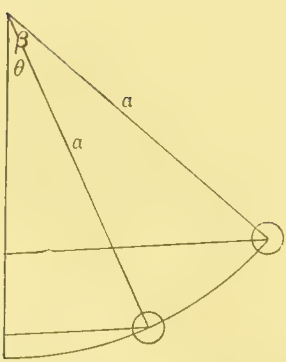
(6) A uniform rod (length $=2a$, radius of gyration about a normal axis through the centre of mass $=k$) is at rest, hanging by a ring attached to its upper end from a smooth fixed horizontal rod. An angular velocity ω is communicated to the hanging rod about an axis perpendicular to the vertical plane through the fixed rod. Find its angular velocity when inclined at an angle θ to the vertical.

Ans. $\left(\dfrac{a\omega^2 - 12g\sin^2(\theta/2)}{a(1+3\sin^2\theta)}\right)^{\frac{1}{2}}$.

(7) Compare the times of oscillation of two pendulums, each of which consists of a massless rod ending in an indefinitely thin massless spherical shell which contains a uniform rigid sphere of the same diameter as the shell, the internal surface of the shell being in one pendulum smooth, in the other perfectly rough, and the dimensions of both pendulums being the same.

Let m be the mass of the sphere, r its radius, and k its radius of gyration about a diameter. Let β be the greatest inclination of the rod to the vertical during an oscillation, θ the inclination at any given instant; and let a be the distance from the centre of the sphere to the point of suspension.

When the rough pendulum falls from inclination β to inclination θ, the potential energy increases by the amount



$$mga(\cos\beta - \cos\theta);$$

and, as the angular velocity of the sphere about its centre of mass is the same as that of the rod at any instant, which we may denote by ω, the kinetic energy increases by the amount $\frac{1}{2}ma^2\omega^2 + \frac{1}{2}mk^2\omega^2$ (442). Hence

$$\tfrac{1}{2}m(a^2+k^2)\omega^2 + mga(\cos\beta - \cos\theta) = 0,$$

and $$\omega^2 = \frac{2ag}{a^2+k^2}(\cos\theta - \cos\beta).$$

When the smooth pendulum falls from inclination β to θ, the potential energy changes by the same amount as in the case of the rough pendulum. The change of kinetic energy however is different. The forces exerted on the sphere by the smooth shell all pass

through its centre. Hence, by the law of the conservation of angular momentum (429), its angular velocity about its centre of mass is constant, and as its value is zero at inclination β it is zero also at inclination θ. Hence, if we denote by ω' the angular velocity of the rod at inclination θ, the kinetic energy of the smooth pendulum increases during the given change of inclination by the amount $\frac{1}{2}ma^2\omega'^2$. We have therefore

$$\tfrac{1}{2}ma^2\omega'^2 + mga(\cos\beta - \cos\theta) = 0,$$

and

$$\omega'^2 = \frac{2g}{a}(\cos\theta - \cos\beta).$$

Now (352, Ex. 5) the angular velocity ω'' of a simple pendulum of length l, acquired in falling from inclination β to inclination θ is such that

$$\omega''^2 = \frac{2g}{l}(\cos\theta - \cos\beta).$$

Hence the lengths of the simple pendulums, which are isochronous with the above rough and smooth pendulums, are $(a^2+k^2)/a$ and a respectively; and hence the times of oscillation through indefinitely small angles are respectively

$$2\pi\sqrt{\frac{a^2+k^2}{ag}} \text{ and } 2\pi\sqrt{\frac{a}{g}}.$$

(8) A pendulum has a bob consisting of a massive block of wood and is at rest. A bullet is fired into the block of wood horizontally, and in a direction perpendicular to the axis of the pendulum. Express the velocity of the bullet in terms of the angle through which the pendulum is deflected. [Such an arrangement is called a *Ballistic Pendulum* and is used for determining the velocities of cannon balls and rifle bullets.]

Let M be the mass of the pendulum with the bullet lodged in it and k its radius of gyration about its fixed axis. Let m be the mass of the bullet, v its velocity, and p the distance of the fixed axis from the line of the bullet's motion. Then mvp is the angular momentum of the system of pendulum and bullet before impact. The block of wood being of great mass (it must be sufficiently massive for this purpose), the ball and block will have the same velocity before the pendulum has been appreciably deflected. The

only external forces acting on the system during the impact are the weights of pendulum and bullet, and in an indefinitely short time these finite forces can produce no change in the angular momentum of the system. Hence the angular momentum immediately after and immediately before impact are the same. Now, if ω is the angular velocity of the pendulum immediately after impact, its angular momentum is $Mk^2\omega$. Hence (453 and 486)

$$mvp = Mk^2\omega.$$

Immediately after impact, as the pendulum is still practically at its lowest position, its kinetic energy has the value $\frac{1}{2}Mk^2\omega^2$. When its angular velocity becomes zero its energy is wholly potential. If θ is the angle through which it has then been deflected, and if h is the distance of the centre of mass from the axis, the increase in potential energy is $Mgh(1-\cos\theta)$. Hence

$$Mgh(1-\cos\theta) = \tfrac{1}{2}Mk^2\omega^2.$$

Hence $$2Mgh\sin^2\frac{\theta}{2} = \frac{1}{2}\frac{m^2v^2p^2}{Mk^2},$$

and $$v = 2\frac{Mk\sqrt{gh}}{mp}\sin\frac{\theta}{2}.$$

(9) A uniform spherical shell whose external radius is n times its internal radius contains a sphere of the same substance, completely filling it. Find the ratio of the space through which the shell would roll from rest in a given time down a perfectly rough inclined plane, if its internal surface were smooth, with the space through which it would roll in the same time if its internal surface were perfectly rough. (The radius of gyration of a sphere about an axis through its centre is $\sqrt{2/5}$ times its radius.)

Ans. $7n^5/(7n^5-2)$.

(10) A uniform rod (length $=2a$) can turn freely about one extremity. In its initial position it is horizontal, and it is projected horizontally with a given angular velocity ω. Show that the least angle θ which it will make with the vertical during its motion is determined by the equation

$$2a\omega^2\cos\theta = 3g\sin^2\theta.$$

500. *Equilibrium of a Rigid Body.*—A rigid body is said to be in equilibrium under the action of forces when its linear and angular momenta are both constant, or, in other words, when its centre of mass has no linear acceleration and the body has no angular acceleration about that point. This state is what we called (444) one of molar equilibrium. It admits of both translation and rotation, but the linear and angular velocities must be constant. If a rigid body be in what we called (444) molecular equilibrium, it may be undergoing translation, but cannot be rotating, and its translational velocity must be uniform.

The conditions of equilibrium (molar) may be obtained from the equations of motion (493)

$$\bar{a}=\Sigma F/\Sigma m, \quad a=\Sigma FP/\Sigma mr^2.$$

That $\bar{a}$ and a may be zero we must have $\Sigma F=0$ and $\Sigma FP=0$. Also, if these conditions are fulfilled, $\bar{a}$ and a must both be zero. Hence the following are both necessary and sufficient conditions of equilibrium (molar), viz., (1) that the algebraic sum of the components in any direction of the external forces be equal to zero, and (2) that the algebraic sum of the moments of the external forces about any axis through the centre of mass be zero.

501. Expressed analytically, these two conditions give us six equations which (415 and 427), if ξ, η, ζ are the co-ordinates, relative to the centre of mass, of the point at which the force acts whose components are X, Y, Z, are as follows:

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma Z=0,$$

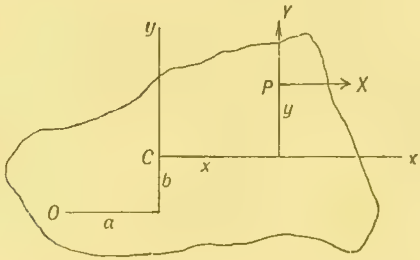
$$\Sigma(Y\xi-X\eta)=0, \quad \Sigma(X\zeta-Z\xi)=0, \quad \Sigma(Z\eta-Y\zeta)=0.$$

If the forces are coplanar (in the xy plane, say), these six reduce to three:

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma(Y\xi-X\eta)=0.$$

502. If a rigid body is in equilibrium, the algebraic sum of the moments of the forces about parallel axes through all points fixed either in or relatively to the body is the same.

Let X, Y be the components of a force acting at P, in the directions of rectangular axes Cx, Cy through the centre of mass C. Let the co-ordinates of P relative to



C be x, y; and let those of C relative to parallel axes through any point O fixed relatively to the body be a, b. Then the algebraic sum of the moments of X, Y about an axis through O perpendicular to the plane of Cx, Cy is

$$Y(x+a)-X(y+b)=Yx-Xy+aY-bX.$$

Hence the algebraic sum of the moments of all the forces about this axis is

$$\Sigma(Yx-Xy)+a\Sigma Y-b\Sigma X=\Sigma(Yx-Xy),$$

since $\Sigma X=\Sigma Y=0$. Hence the sum of the moments of the forces about an axis through any point O is equal to that about a parallel axis through the centre of mass.

In the above, the system is supposed to be in one plane. The result will obviously be the same, however, if the force acting at P have a component Z in the direction perpendicular to Cx and Cy, if the co-ordinates of P are x, y, z, and if those of C are a, b, c.

503. Hence the following are necessary and sufficient conditions of equilibrium (molar): (1) that the algebraic

sum of the components, in any direction, of the external forces be equal to zero; (2) that the algebraic sum of the moments of the external forces about any axis through any point fixed relatively to the body be zero.

504. In special cases the above conditions take special forms:

(*a*) *Body under two forces.*—If the rigid body be in equilibrium under two external forces only, they must obviously be equal and opposite and have the same line of action.

505. (*b*) *Body under Parallel Forces.*—One of them must obviously be equal and opposite to the resultant of all the rest and must have the same line of action. Hence it must be equal to their sum, and its line of action must pass through the point called their centre (472).

506. (*c*) *Body under three Non-Parallel Forces.*—If a body be in equilibrium under three non-parallel forces, their lines of action must be coplanar and must all pass through one point.—If two of the forces are in one plane, the third must be in this plane also; for otherwise it would have an unbalanced component normal to the plane. If no two are in one plane, a plane through one will intersect the action lines of the other two, which will therefore have components normal to it. That these components may balance one another, the plane must intersect their action lines in the same point. These two forces therefore, and consequently by the above all three, must be in the same plane. Hence, generally the three forces must be coplanar. Again, about the point of intersection of the lines of action of any two of these forces, these two can have no moment. Hence also the third can have no moment about it, and consequently

the line of action of the third force must pass through the intersection of the lines of action of the other two. If therefore a body be in equilibrium under three non-parallel forces, these forces must be coplanar and must all pass through one point.

Hence the conditions of equilibrium of a rigid body under three forces are exactly the same as those applicable to a particle under three forces. Hence the results of 325, (*c*), (*d*), and (*e*), deduced in the case of a particle, apply also to the equilibrium of a rigid body.

507. *Examples.*

(1) A uniform right-angled-triangular plate is suspended by a string from the right angle. Show that its sides make the same angles with the vertical as they do with the base.

[The only forces acting are the stress in the string and the weight of the triangle. Hence the stress and therefore the string must be vertical, and their directions must pass through the centre of mass of the triangle, and consequently through the centre of the base.]

(2) A hemisphere and a cone are fastened together by their bases which are equal, and the body so formed rests in equilibrium on a horizontal plane in whatever position it may be placed. Show that its centre of mass coincides with the common centre of their bases.

(3) A body of mass m hangs from the edge of a uniform hemisphere (mass $=M$, radius $=r$, distance of centre of mass from centre $=3r/8$), which rests with its convex surface on a smooth horizontal plane. Find the inclination θ of the axis of the hemisphere to the vertical.

Ans. $\theta=\tan^{-1}\dfrac{8m}{3M}$.

(4) Two men carry 3 cwt. by a pole 8 ft. long, which they support at the ends. If the body be hung 1 ft. from the middle of the pole, what forces are exerted by the men?

Ans. Forces equal to the weights of $1\frac{7}{8}$ and $1\frac{1}{8}$ cwt.

(5) AB is a heavy straight rod or lever of length l. When a body of weight W is suspended from A, the rod balances about a

fulcrum (or fixed point) at a distance a from A, and when the same body is suspended from B, it balances about a fulcrum at a distance b from B. Find (1) the distance of the centre of mass of the rod from A, and (2) the weight of the rod.

Ans. (1) $\frac{al}{a+b}$; (2) $\frac{(a+b)W}{l-a-b}$.

(6) The mass of a window-sash, 3 ft. wide, is 5 lbs., and that of each of the "weights" attached to the cords 2 lbs. If one of the cords be broken, at what distance from the middle of the sash should the hand be placed to raise it with the least effort?

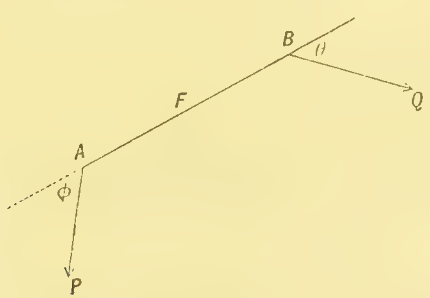
Ans. 1 ft.

(7) A rigid rod ABC, suspended by the point A, is composed of two pieces rigidly connected at B, and inclined at a right angle to one another. Show that if a and c are the lengths of the arms AB and BC respectively and θ the inclination of AB to the vertical,

$$c^2 \cot \theta = a^2 + 2ac.$$

(8) A heavy uniform bar lies with two-thirds of its length on a smooth horizontal table. Show that a body weighing more than half as much as the bar would, if suspended at the free extremity, pull it over.

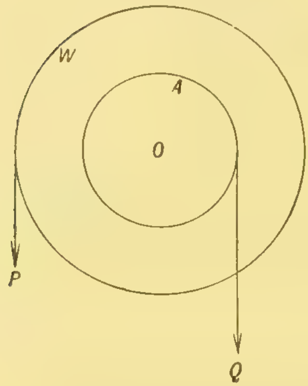
(9) Two forces P and Q act at the ends of a straight weightless lever AB, their directions being inclined to AB at the angles ϕ and



θ respectively. Find the position of the fulcrum F (*i.e.*, find what point F must be fixed), that equilibrium may be maintained.

Ans. $AF = \frac{Q \cdot AB \cdot \sin \theta}{P \sin \phi + Q \sin \theta}$.

(10) The radii of the wheel (W) and the axle (A) of the simple machine called the Wheel and Axle (254, Ex. 5) are R and r respectively. Find the force P exerted through a string coiled round the wheel which will balance a force Q exerted through a string coiled round the axle (the axle being smooth).



Since the sum of the moments about the fixed axis O must be zero, $PR - Qr = 0$ and $P = Qr/R$.

(11) Find the conditions that must be fulfilled that a Balance may be stable and sensitive. (See 498, Ex. 4.)

The beam without the pans will be in stable equilibrium (450), if G (fig. of 498, Ex. 4) be vertically below O, in which case BC will be horizontal. If the centre of mass of the beam be at O, the beam without the pans will be in neutral equilibrium. With pans of equal mass the beam will be in stable equilibrium, with BC horizontal, provided (1) that G and BC are both below O, or (2) that if G be above O, BC be sufficiently far below it, or (3) that if BC be above O, G be sufficiently far below it.

When the balance is in equilibrium, T and T' are vertical and equal to mg and $m'g$ respectively. As the sum of the moments of the forces acting on the beam about O must be zero, we have, if m and m' are the masses of the pans with their contents, and θ the inclination of BC to the horizon,

$$mg(AC\cos\theta - OA\sin\theta) - m'g(AB\cos\theta + OA\sin\theta) - Mg\,.\,OG\,.\sin\theta = 0.$$

Hence $$\tan\theta = \frac{(m-m')AB}{(m+m')OA + M\,.\,OG}.$$

The greater the angle θ for a given value of $m - m'$ the greater is the sensitiveness of the balance. Hence for sensitiveness the mass of the beam, the load (*i.e.*, the total mass in both pans) and the distance of the axis of the beam both from its centre of mass and from the line joining the points of suspension of the pans, must be as small as possible, and the distance between the points of suspension of the pans must be as great as possible. Except with regard to

the mass of the beam, the conditions for sensitiveness and for quickness of motion (498, Ex. 4) are antagonistic. Hence in all balances the mass of the beam is made as small as is consistent with sufficient rigidity, and a compromise is struck between the demands of sensitiveness and of quickness of motion with regard to length of arm, etc.

If the line BC pass through O, $OA=0$. Hence

$$\tan\theta = \frac{(m-m')AB}{M.OG}.$$

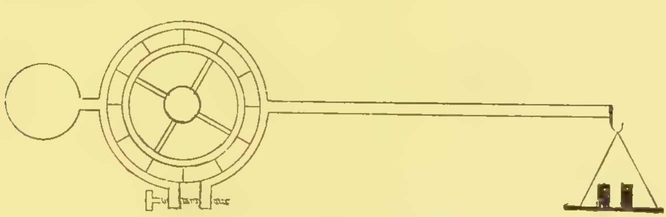
In this case, therefore, the sensitiveness is independent of the load.

(12) m_1, m_2, are the apparent values of the mass of a body when weighed successively in both pans of a balance which has its three suspension points in a straight line. (a) If its pans are equal and its arms unequal, show that the real mass of the body is $\sqrt{m_1 m_2}$. (b) If its arms are equal but the pans unequal, show that the difference of the masses of the pans is $\frac{1}{2}(m_1 - m_2)$.

(13) The beam of a false balance (*i.e.*, one having unequal arms) is 3 ft. long. If a certain body is placed in one scale it weighs 4 lbs., if in the other 6 lbs. 4 oz. Find (a) the real mass of the body and (b) the lengths of the arms.

Ans. (a) 5 lbs.; (b) 1 ft. 4 in. and 1 ft. 8 in.

(14) The shaft of a steam engine carries a strong wheel (radius $=r$) with a flat rim. An iron strap lined with blocks of wood is



fitted round it, and presses the blocks against the flat rim of the wheel. A rod is attached to the strap, and carries at its end (distant l from centre of shaft) a pan for standards of mass. When

the shaft is making n revolutions per second, and doing no work except that of overcoming the friction of the strap, the rod is maintained in a horizontal position by putting standards, of weight W, into the pan. Find the rate at which the engine is working. [This arrangement is called the *Friction Brake Dynamometer*. It should be called an ergometer, as it is not force that we measure by it, but rate of work done.]

The rigid system, consisting of strap, rod, and wooden blocks, is in equilibrium under its own weight, the weight of the standards, and the friction of the rotating wheel. The rod is always counterpoised, so that its own weight passes through the shaft of the engine, which is the fixed axis of the rigid system under consideration. Hence if F is the friction of any small element of the surface of the wheel, we have

$$\Sigma Fr - Wl = 0.$$

Hence $$Wl = \Sigma Fr = r\Sigma F.$$

The work done against friction during n revolutions of the shaft at each element of the surface of contact is $2\pi rnF$. Hence the whole work done against friction per second is

$$2\pi nr\Sigma F = 2\pi n Wl.$$

Hence the rate at which the engine is working is expressed in terms of n and W.

(15) A uniform thin triangular plate is supported in a horizontal position by three props at its angular points. Show that the pressures on the props will be equal.

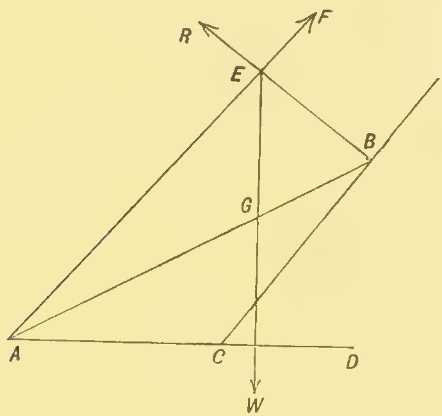
[As there is equilibrium, the moment of the force exerted by any prop about an axis through the points at which the other props touch the plate must be equal to the moment of the weight about the same axis.]

(16) A triangular plate is hung with its plane horizontal by three vertical chains, attached to the middle points of its sides. What must its mass be that a ton may be placed anywhere on it without tilting it?

Ans. 3 tons.

(17) A rod AB hinged at one end so that it can move in a vertical plane rests with the other end on a smooth inclined plane, whose line of intersection with the horizontal plane is perpendicular to the plane of AB's motion. Find the force exerted on it by the hinge and the reaction of the plane.

Let AB be the rod hinged at A and resting with the end B on the inclined plane BC. Let ACD be the horizontal plane, and let the angles BCD, BAC, be θ and ϕ respectively.



The rod is acted upon by three forces—its weight W acting vertically through G its centre of mass, the normal reaction R of the inclined plane, and the force F exerted by the hinge. Hence (506) the force F must pass through the intersection E of W and R. The direction of F is known if the angle $EAB(\psi)$ is known. Now,

$$\frac{AG}{GE}=\frac{\sin AEG}{\sin\psi}=\frac{\sin(\frac{1}{2}\pi-\phi-\psi)}{\sin\psi}=\frac{\cos(\phi+\psi)}{\sin\psi},$$

and $$\frac{GB}{GE}=\frac{\sin GEB}{\sin GBE}=\frac{\sin\theta}{\sin(\frac{1}{2}\pi+\phi-\theta)}=\frac{\sin\theta}{\cos(\theta-\phi)}.$$

Hence $$\frac{AG}{GB}=\frac{\cos(\theta-\phi)}{\sin\theta}\cdot\frac{\cos(\phi+\psi)}{\sin\psi};$$

and $$\psi=\cot^{-1}\left(\frac{AG}{GB}\cdot\frac{\sin\theta}{\cos\phi\cos(\theta-\phi)}+\tan\phi\right).$$

For the magnitude of F we have (506)

$$W : F = \sin \widehat{FR} : \sin \widehat{WR}$$
$$= \sin AEB : \sin GEB$$
$$= \sin(\tfrac{1}{2}\pi - \phi - \psi + \theta) : \sin\theta$$
$$= \cos(\phi + \psi - \theta) : \sin\theta.$$

Hence $$F = W\frac{\sin\theta}{\cos(\phi+\psi-\theta)}.$$

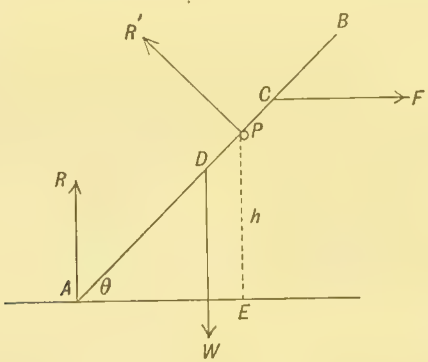
For the magnitude of R we have similarly

$$W : R = \sin \widehat{FR} : \sin \widehat{WF};$$

and hence $$R = W\frac{\cos(\phi+\psi)}{\cos(\phi+\psi-\theta)}.$$

(18) A uniform rod AB rests over a smooth peg at P and with its end A on a smooth horizontal plane, being acted on at C by a horizontal force F in the vertical plane through the rod. For a given value of F, find the position of the rod, and find the reactions of the plane and peg in this position.

Four forces act on the rod, its weight W acting through its middle point D, the reactions R and R' of the horizontal plane and peg, normal to the plane and rod respectively, and the force F.



R, W, and F are in a vertical plane through the rod. Hence R' is in that plane also.

The position of the rod is determined if the angle (θ) between it and the horizontal plane is known. Let the distance of P from that plane be h, the length of the rod l, and the distance of C from B, c.

For equilibrium : (1) The sum of the vertical components of the external forces must be zero. Hence

$$R - W + R' \cos\theta = 0.$$

(2) The sum of the horizontal components of the external forces must be zero. Hence $F - R' \sin\theta = 0.$

(3) The sum of the moments of the external forces about any point A must be zero. Hence

$$R'h/\sin\theta - F(l-c)\sin\theta - \tfrac{1}{2}Wl\cos\theta = 0.$$

These three equations involve only the three unknown quantities θ, R, R', and are therefore sufficient to determine them.

For the resolution of the acting forces we select those directions, and for the expression of their moments we select those points, which will give the simplest equations. Thus in the above example the equations obtained by resolving horizontally and vertically are simpler than those which would be obtained by a resolution in and normal to the direction of AB, because the forces R and W have no horizontal component, and F has no vertical component, while R' is the only force with no component, in the direction of AB, and all the forces have components in the direction normal to it. Similarly it is better to take moments about A, D, P, or C than about any other point, because in each of these cases one of the acting forces will have no moment. If the force F acted at D, that would be the best point to take moments about, as in that case two forces would be excluded from the equation of moments.

If we do not wish to determine all the unknown quantities, we may select the directions and the point referred to above, with a view to the exclusion from the equations, of the quantity or quantities which we do not wish to determine. Thus if we wish in the above example to find only R and θ, we select the direction of AB for resolution of forces, and we obtain

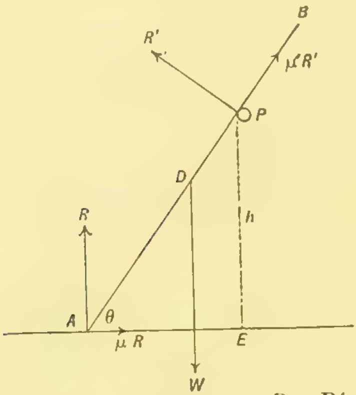
$$(R - W)\sin\theta + F\cos\theta = 0\,;$$

and selecting the point P for the summation of moments, we obtain

$$W\cos\theta\left(\frac{h}{\sin\theta} - \tfrac{1}{2}l\right) - R\cos\theta\,.\,\frac{h}{\sin\theta} - F\sin\theta\left(l - c - \frac{h}{\sin\theta}\right) = 0.$$

These two equations are sufficient to determine the two unknown quantities, R and θ.

(19) The horizontal plane and the peg of Ex. 18 being rough and the force F being withdrawn, find the position of equilibrium when the rod is on the point of slipping down.



The roughness of the plane and peg introduces two new forces acting at A and at the peg, in the directions of AE and AB, and equal to μR and $\mu' R'$ respectively (μ being the coefficient of friction between the plane and the rod, μ' that between the peg and the rod). Resolving horizontally and vertically, and taking moments about A as before, we obtain the three equations :—

$$\mu R - R' \sin\theta + \mu' R' \cos\theta = 0,$$

$$R - W + R' \cos\theta + \mu' R' \sin\theta = 0,$$

$$R'h/\sin\theta - \tfrac{1}{2} Wl \cos\theta = 0,$$

which are sufficient to determine R, R', and θ.

(20) A uniform straight rod moveable about its lower extremity leans against a vertical wall and makes an angle of 45° with the horizon. Show that the force exerted by the wall on the rod is equal to half the weight of the rod.

(21) A uniform beam AB (weight $= W$) rests with one end A on a smooth horizontal plane, and the other B against a smooth vertical plane, the vertical plane through the beam intersecting at right angles the former in the line AC and the latter in BC. The beam is attached to the point C by a string AC. Find (a) the tension in AC, (b) the reaction of the horizontal plane, (c) the reaction of the vertical plane.

Ans. (a) $\tfrac{1}{2} W \cot BAC$; (b) W ; (c) $\tfrac{1}{2} W \cot BAC$.

(22) If the string in Ex. 21 is attached to a point E in the beam between A and its middle point, show that the tension will be

$$\tfrac{1}{2} W \frac{\cos BAC}{\sin(BAC - ECA)}.$$

Hence show that the length and point of attachment of the string

must be such that the angle BAC is greater than ECA in order that equilibrium may be possible.

(23) If in Ex. 21 the horizontal and vertical planes be rough (co-efficient of friction in both cases $=\mu$), and if there be no string, find the position of equilibrium.

Ans. Angle $BAC=\tan^{-1}\dfrac{1-\mu^2}{2\mu}$.

(24) A rod (weight $=W$; distance of centre of mass from lower end $=l$) rests upon a smooth prop, with the lower end pressing against a smooth vertical wall (distance from prop $=d$), the vertical plane through the rod being at right angles to the wall. Find (*a*) the position of equilibrium; (*b*) the reaction of the prop; (*c*) the reaction of the wall; and show that equilibrium is impossible unless l be greater than d.

Ans. (*a*) Inclination of rod to wall $=\sin^{-1}(d/l)^{\frac{1}{3}}$; (*b*) $W(l/d)^{\frac{1}{3}}$; (*c*) $W(l^{\frac{2}{3}}-d^{\frac{2}{3}})^{\frac{1}{2}}/d^{\frac{1}{3}}$.

(25) A uniform rod rests with one end pressing against the inner surface of a fixed smooth hemispherical bowl (radius $=r$) whose rim is horizontal, and with the other projecting beyond the rim. It is inclined 30° to the horizon. Find its length.

Ans. $4r/\sqrt{3}$.

(26) A sphere (weight $=W$) rests upon two inclined planes (inclinations to the horizon $=\theta$ and θ'). Find the reactions of the planes.

Ans. $W\sin\theta'/\sin(\theta+\theta')$, and $W\sin\theta/\sin(\theta+\theta')$ respectively.

(27) One extremity A of a beam AB (length $=l$, distance of centre of mass from $B=n$ times its distance from A) rests against a rough vertical wall (angle of repose $=\epsilon$), and a cord tied to the other extremity B is fastened at a point in the wall above A, the vertical plane through the rod being perpendicular to the wall. Show that, if the rod is to be horizontal, the length of the cord must be $\dfrac{l}{n}\sqrt{n^2+\tan^2\epsilon}$.

(28) A uniform heavy rod, 2 ft. long, is hung up to a peg by means of two strings tied to its ends, the lengths of the strings being 1 ft. and $\sqrt{3}$ ft. respectively. Show that, when the rod is

in equilibrium, it will make an angle of 30° with the horizon, and the tension of the shorter string will be equal to half the weight of the rod.

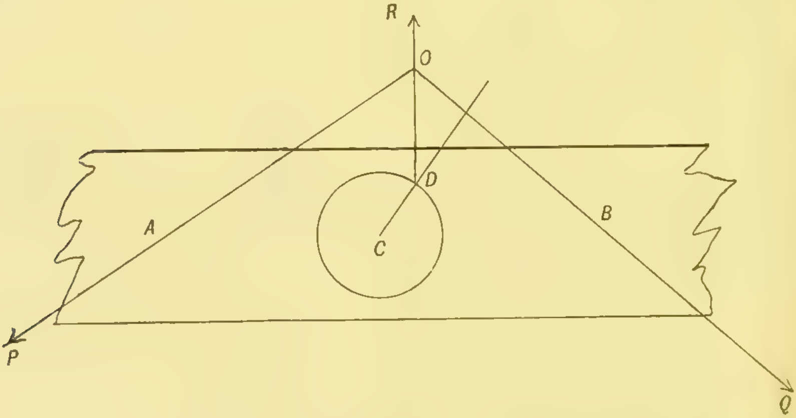
(29) A uniform heavy rectangular trap-door is moveable about one edge as a hinge-line. To the middle point A of the opposite edge is attached a string which passes over a smooth pulley at the point occupied by A when the door is horizontal, and sustains a body of weight w. If W be the weight of the door, show that the inclination of the door to the horizon is given by the equation

$$\cos^2 \tfrac{1}{2}\theta - \frac{w}{W} \cos \tfrac{1}{2}\theta - \tfrac{1}{2} = 0.$$

(30) A carriage wheel (weight $= W$, radius $= r$) rests upon a level road. Show that the force necessary to draw it over an obstacle of height h is $W\sqrt{h(2r-h)}/(r-h)$.

(31) A heavy uniform sphere hangs from a peg by a string, the length of which is equal to the radius, and rests against another peg, vertically below the former, the distance between the two being equal to the diameter. Show that the tension of the string is equal to the weight, and the reaction of the peg to half the weight, of the sphere.

(32) A beam or lever is moveable about a fixed rough cylindrical axle (radius $= r$, angle of repose $= \epsilon$), which very nearly fills the



hole in the beam through which it passes. Find the relation between two forces P and Q acting on the beam at given points

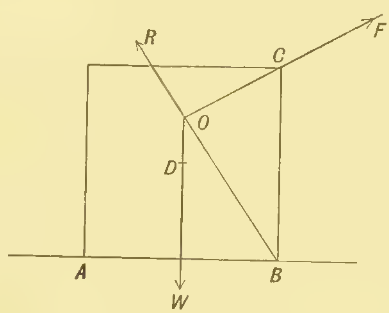
A and B and in given directions in a plane perpendicular to the axle, when the beam is on the point of moving.—Let θ be the inclination of P and Q, and let p, q be their distances from C the centre of the axle. Produce P and Q to meet in O. Then R, the reaction of the axle, must pass through O. Since the axle only nearly fills the socket there is contact, at any instant, only along a single line. If this line is represented in the diagram by the point D, DO will be the direction of R and will be inclined to CD produced at the angle ϵ. Hence the distance of R from C is $r \sin \epsilon$. For equilibrium therefore we have

$$Qq = Pp + Rr \sin \epsilon.$$

Hence
$$Qq = Pp + r \sin \epsilon \sqrt{P^2 + Q^2 + 2PQ \cos \theta},$$

which is the required relation between P and Q.

(33) A heavy homogeneous cubical block rests on a rough horizontal plane, and a force is exerted on it by means of a string attached to the middle point of one of the upper edges, the string and the centre of mass being in the same vertical plane. The force being gradually increased, find the nature of the initial motion of the block.—Let ABC be the plane in which all the forces act, and let F's line of action be above the centre of mass D. Then the initial motion of the block will clearly be either a sliding in the direction of AB or a turning about the edge B. For $F=0$, the reaction of the plane is normal to AB; but, as F is gradually increased, the reaction (328) becomes gradually more and more inclined to the normal, passing, since there is equilibrium, through the intersection O of F and W. If the cube turn about B, the reaction must pass through B. If therefore it is on the point of turning about B, the line of the reaction must be BO. Hence, if the friction is such that the angle CBO is less than the angle of repose, the initial motion will be a turning about B. If however CBO is greater than the angle of repose, slipping will be the initial motion;



for the block will begin to slip when the reaction is inclined to the normal at the angle of repose. Let θ be the inclination of F to AB, ϵ the angle of repose, and a the edge of the cube, then the condition for initial turning is

$$\tan \epsilon > \tan CBO$$

$$> \frac{a}{2} \Big/ \left(a - \frac{a}{2} \tan \theta\right)$$

$$> 1/(2 - \tan \theta).$$

Hence also the condition for initial sliding is

$$\tan \epsilon < 1/(2 - \tan \theta).$$

If F's line of action is below D, the possible initial motions are sliding in the direction AB and turning about the edge A. Show that the condition of initial turning about A is

$$\tan \epsilon > 1/(\tan \theta - 2).$$

(34) A homogenous right cone (vertical angle$=2\theta$) is placed with its base on a rough inclined plane (coefficient of friction$=\mu$), whose inclination is gradually increased. Show that, if $\mu > 4 \tan \theta$, the initial motion of the cone will be tumbling, and if $\mu < 4 \tan \theta$, its initial motion will be sliding.

(35) A rectangular block is placed with one of its edges horizontal on a rough inclined plane. Show that, if a is the length of the edge of the block which is perpendicular to the plane, and b the length of the other non-horizontal edge, and if μ is the coefficient of friction, the initial motion will be one of tumbling, provided $\mu > b/a$, and of sliding, provided $\mu < b/a$.

(36) A rectangular block, weighing 20 lbs., with a square base 8 inches in side, is set up on a level table, and it is found that a horizontal force equal to the weight of 5 lbs., if applied below a certain point, is just able to make it slide, while, if it is applied above that point, the block topples over. Find (a) the position of this critical point, and (b) the coefficient of friction between the block and the table.

Ans. (a) 16 in. from the base; (b) 0·25.

508. *Equilibrium of a System of Rigid Bodies.*—If two or more rigid bodies be connected by strings, rods, joints, etc., the system is said to be in equilibrium provided (1) the system behave as a rigid body and be in molar equilibrium, or (2) each body of the system be in molar equilibrium.

(1) If the system behave as a rigid body, its parts not moving relatively to one another, the necessary and sufficient condition of equilibrium is (500) the satisfaction of the equations: $\Sigma F = 0$, $\Sigma FP = 0$, the forces involved being those external to the system only.

(2) The necessary and sufficient condition for the equilibrium of each body of the system is the satisfaction for each body of the equations: $\Sigma F = 0$, $\Sigma FP = 0$, the forces involved in the equations including forces external to the body to which they apply, and therefore in general some forces which are internal to the system.

509. *Examples.*

(1) A body is to be supported by means of the system of smooth pulleys represented in Fig. 1, p. 426. The weight of the body being W, and that of the block (254, Ex. 6) w, find the force F which must be applied at the end of the cord.

The pulleys being smooth, the stress throughout the whole string is F (391). Hence, if there are n sheaves in each block, the lower block is acted upon by $2n+2$ forces, $2n$ having each the magnitude F and an upward direction, and two the magnitudes W and w respectively and downward directions. If the directions of all are taken to be vertical, we have therefore

$$2nF = W + w.$$

The ratio W/F is called the mechanical advantage of the system of pulleys. If $w=0$, it has in this system the value $2n$.

(2) Find the mechanical advantage of the system of smooth weightless pulleys represented in Fig. 2, there being n moveable pulleys.

Ans. 2^n.

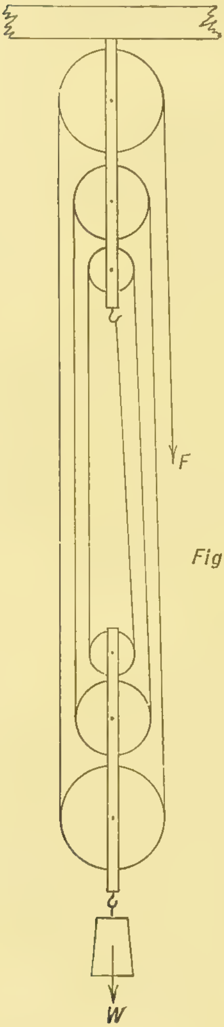


Fig 1

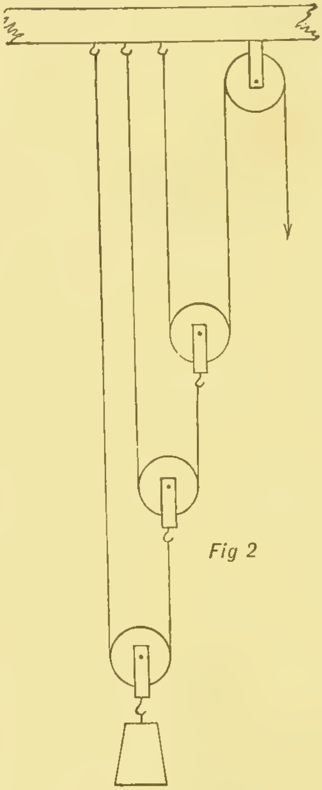


Fig 2

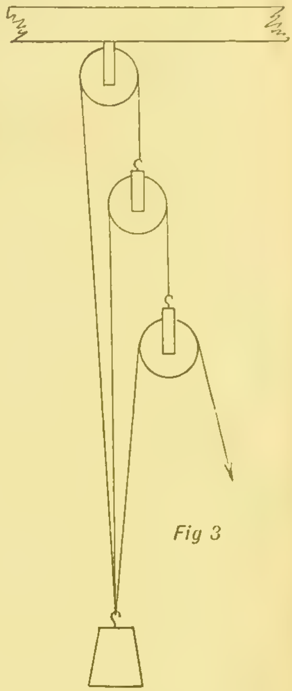


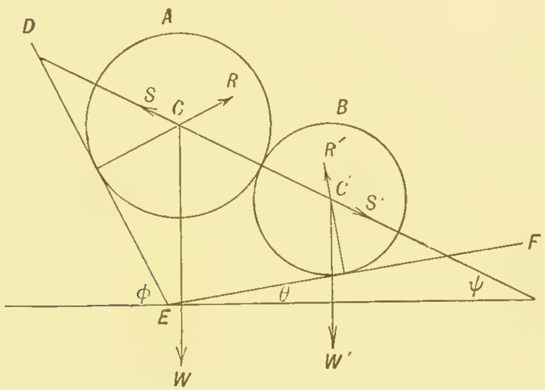
Fig 3

(3) Find the mechanical advantage of the system of smooth weightless pulleys represented in Fig. 3, there being n pulleys, and the ropes being so long that they may all be considered vertical.

Ans. $2^n - 1$.

(4) A system of smooth weightless pulleys, like that of Ex. 1, but with only one moveable pulley and with a body hanging at F, is in equilibrium. Show that if the body supported by the moveable pulley have its mass doubled, and the other its mass halved, the tension in the string will be unaltered.

(5) Two smooth spheres rest on two smooth inclined planes and press against each other. Determine their position and the magnitudes of the reactions.—Let A and B be the spheres, C and C' their centres, DE and EF the inclined planes of inclinations ϕ and θ



respectively. Each sphere is acted upon by three forces—its weight (W, W'), the normal reaction of the plane with which it is in contact (R, R'), and the normal reaction of the other sphere (S, S'). As each sphere is acted upon by three forces only, these three must in each case be in the same plane, but as the lines of action of S and S' coincide with the line CC', W, W', S and S' are in the same plane. Hence all six forces are in the same plane, which is consequently a vertical plane and perpendicular to both inclined planes. Let that be the plane of the diagram. The positions of the spheres are determined by the angle ψ, the inclination of CC' to the horizon.

For the equilibrium of A we have, resolving in the direction of DE

$$W\sin\phi - S\cos(\phi-\psi)=0,$$

and resolving in a perpendicular direction,

$$R - W\cos\phi - S\sin(\phi-\psi)=0.$$

For the equilibrium of B we have, similarly,

$$W' \sin \theta - S' \cos(\theta + \psi) = 0,$$

and
$$R' - W' \cos \theta - S' \sin(\theta + \psi) = 0.$$

As $S = S'$, the above four equations are sufficient to determine all the unknown quantities involved, viz., R, R', S, and ψ. To determine ψ only, the first and third equations are sufficient.

If we regard the whole system as a rigid body, S and S' become internal forces, and may be left out of account. Equating to zero (1) the vertical and (2) the horizontal components of external forces, we find

$$W + W' - R \cos \phi - R' \cos \theta = 0,$$

and
$$R \sin \phi - R' \sin \theta = 0.$$

Also equating to zero the sum of the moments of external forces about C, we have,

$$W' \cos \psi - R' \cos(\theta + \psi) = 0.$$

We have thus three equations for the determination of the three unknown quantities R, R', and ψ.

(6) Two smooth spheres of equal radius r and weight W are placed inside a uniform thin hollow cylinder (radius $= r' < 2r$) which is open at both ends and rests with one end on a horizontal table. What must the weight of the cylinder be that it may not upset?

Ans. $2W(r' - r)/r'$.

(7) A smooth sphere (weight $= W$) rests upon two equally inclined planes (inclination $= \alpha$) which are placed on a smooth horizontal table, and are prevented from sliding apart by a horizontal string which binds them together. Find the tension in the string.

Ans. $\frac{1}{2} W \tan \alpha$.

(8) Of four equal smooth spheres (weight of each $= W$) three rest in contact on a smooth horizontal plane, and the fourth is placed upon them. Find the horizontal force which must be applied to each of the three to preserve equilibrium.

Ans. $W/3\sqrt{2}$.

(9) A heavy uniform smooth beam (weight $= w$, length $= 2a$) is moveable in a vertical plane about a smooth hinge at one end. A heavy smooth sphere (weight $= W$, radius $= r$) is attached to the

hinge by a string (length $=l$), and the two bodies rest in contact. Obtain equations for determining the inclination θ of the string to the vertical, the inclination ϕ of the beam to the vertical, the reaction S of the hinge on the beam, and the stress R between the beam and the sphere.

Ans.
$$W(l+r)\sin\theta = wa\sin\phi,$$
$$(l+r)\sin(\theta+\phi)=r,$$
$$R\cos(\theta+\phi)=W\sin\theta,$$
$$(S^2-w^2)\cos^2(\theta+\phi)=W^2\sin^2\theta-2wW\sin\theta\sin\phi\cos(\theta+\phi).$$

(10) A uniform heavy rod (weight $=W$, length $=2l$) connects the centres of two equal heavy wheels (radius $=r$), which rest on a rough inclined plane (coefficient of friction $=\mu$) in a vertical plane, which is a plane of greatest slope of the inclined plane, the lower wheel being locked. Find the greatest inclination of the plane that will admit of equilibrium.

Ans. $\tan^{-1}\dfrac{\mu l}{2l-\mu r}$.

(11) Three horizontal weightless levers, AEB, BFC, CGD, the fulcrums of which are at E, F, G, act upon one another perpendicularly, the first and second at B and the second and third at C. They are kept in equilibrium by bodies hanging from the points A, D, and weighing W and $2W$ respectively. AE, EB, BC, CG, GD are 1, 2, 7, 2, 3 ft. respectively. Find (a) the position of F, and (b) the reaction of the fulcrum at F.

Ans. (a) $FC=1$ ft.; (b) $7W/2$.

(12) Two beams whose weights are proportional to their lengths (9 and 7 ft.) rest with their lower ends in contact on a smooth horizontal plane, and their upper ends leaning against two smooth vertical and parallel walls 10 ft. apart. Show that if θ and θ' are the respective inclinations of the beams to the horizon, we have, to determine them,

$$7\tan\theta=9\tan\theta', \text{ and } 9\cos\theta+7\cos\theta'=10.$$

(13) Two uniform straight rods of equal length rest with their lower ends on a rough horizontal plane (coefficient of friction $=\mu$)

and their upper ends in contact, and are on the point of slipping. Find the common inclination to the horizon.

Ans. $\tan^{-1}(1/2\mu)$.

(14) Two bars which are connected by a smooth hinge or joint are in equilibrium. Investigate generally its reactions on the bars.

(*a*) If the hinge pin is rigidly connected with one of the bars, the reactions between the bars are obviously equal and opposite, their magnitude and direction depending upon the external forces acting upon the bars.

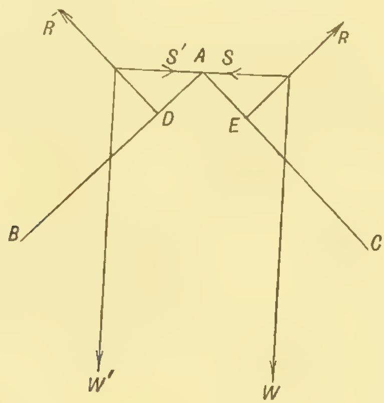
(*b*) If the hinge pin is distinct from both bars, and if no external forces act on the pin (which condition requires either that the weight of the pin should be negligible, or that it should be neutralized by an equal and opposite external force), its reactions on the bars must be equal and opposite. For the pin is in equilibrium under the two forces exerted upon it by the bars at the points or rather lines of contact, and as these forces must therefore be equal and opposite, the reactions of the pin on the bars must also be equal and opposite. If however the pin is acted upon by an external force, the forces exerted upon it by the bars will not have the same line of action, and its reactions on the bars will therefore also have different lines of action.

(15) In a system of jointed thin bars, in which the hinge-pins are distinct from the bars, if the external forces act only on the hinge-pins (this condition implies that the weights of the bars are negligible), the reactions of the pins on the bars will be in the directions of the bars.

For in that case any bar is acted upon by two forces only, the reactions of the hinges at its ends. These forces must therefore be equal and opposite, and their lines of action must consequently be the direction of the bar.

(16) Two equal uniform rods, equally inclined to the horizon, and connected by a smooth hinge at their higher ends, pass through two small fixed rings in a horizontal line. Find the inclination of either rod, when the system is in equilibrium, and the reactions of the hinge on the rods.

Let AB, AC be the two rods hinged at A; D and E the small rings. The rods are acted upon by their weights (W, W') and the



reactions of the rings (R, R') and of the hinge (S, S'). The reactions at A must be in the same straight line, must pass through the intersection of the lines of action of the weight and of the reaction of the ring in the case of each rod, and must therefore be horizontal. Hence the centres of mass of the rods must be between the rings and their lower end points.

Let l be the length of each rod, d the distance between the two rings, and θ the inclination of each rod to the horizon. Resolving the forces acting on AC in the direction of AC, we have,

$$W \sin\theta - S\cos\theta = 0,$$

and, taking moments about E, we have

$$W\cos\theta(l - d/\cos\theta) - Sd\tan\theta = 0.$$

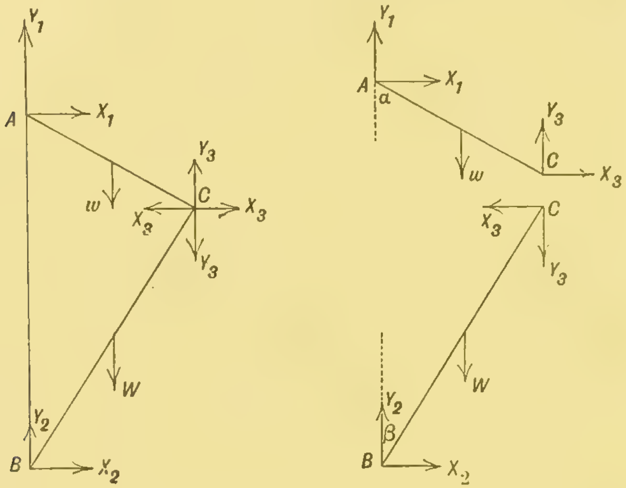
From these equations we may obtain both S and θ.

(17) Three rods jointed together at their extremities, are laid on a smooth horizontal table, and horizontal forces are applied at their middle points perpendicularly to them. Show that if these forces produce equilibrium, the stresses at the joints will be equal and their directions will touch the circle circumscribing the triangle. (See 475, Ex. 1.)

(18) Two uniform rods AC, BC (weights $= w$ and W) are connected by a smooth hinge at C, their other ends A and B being fastened to

fixed hinges in a vertical line. Find the reactions of the hinges on the bars.

Let X_1, Y_1 and X_2, Y_2 be the horizontal and vertical components of the reactions on AC and BC at A and B respectively. (It does not matter in what direction, up or down, we draw Y_1, Y_2 or whether we draw X_1, X_2 to the right or left. If the actual reactions have components in directions opposite to those assumed in



the diagram, the values of X_1, Y_1, etc., as the case may be, will be found negative.) Let X_3, Y_3 be the components of the reaction of the hinge at C on the rod AC. Then, since the hinge-pin is not acted on by external forces, its weight being negligible, its reaction on BC will have components $-X_3$, $-Y_3$. As the four forces shown in the diagram as acting at C act two on AC and two on BC, it is often advisable to draw a special diagram for each bar. Such diagrams are shown above. The equations of equilibrium may be written down by their aid without danger of inserting BC's forces in AC's equation. Thus for the equilibrium of AC we have

$$X_1+X_3=0,$$

$$Y_1+Y_3-w=0,$$

and taking $AC=2a$, and calling its inclination to AB, α,

$$2a(X_1\cos\alpha+Y_1\sin\alpha)-wa\sin\alpha=0.$$

And for the equilibrium of BC, calling its length $2b$, and its inclination to AB, β, we have

$$X_2 - X_3 = 0,$$

$$Y_2 - Y_3 - W = 0,$$

$$2b(Y_2 \sin\beta - X_2 \cos\beta) - Wb \sin\beta = 0.$$

We have thus six equations which are sufficient to determine the six unknown components of the reactions.

(19) Two uniform beams AC and CB (weights $= W$ and W' respectively) connected at C by a smooth joint are placed in a vertical plane, their extremities A, B being connected by a string and resting on a smooth horizontal plane (inclinations of AC, CB to the horizon $= \alpha$ and α' respectively). Find (a) the tension in the string, and (b) the reaction at the joint.

Ans. (a) $\dfrac{W + W'}{2(\tan\alpha + \tan\alpha')}$;

(b) magnitude $= \dfrac{\sqrt{(W + W')^2 + (W'\tan\alpha - W\tan\alpha')^2}}{2(\tan\alpha + \tan\alpha')}$;

line of action inclined to horizon at $\tan^{-1}\left(\dfrac{W'\tan\alpha - W\tan\alpha'}{W + W'}\right)$.

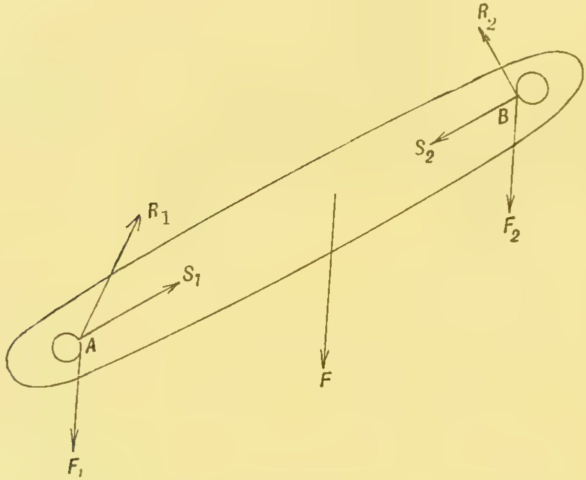
(20) Two rods AB, CD are connected by smooth hinges at A, D to two fixed points in the same horizontal line, and at B, C, also by smooth hinges to the ends of a rod BC. Show that if all three rods are of equal length, and if either AB or CD is inclined $\alpha°$ to the horizon, the inclination to the horizon of the reaction of the joint at its lower end will be $\tan^{-1}(\frac{1}{2}\tan\alpha)$.

(21) A plane polygonal frame, composed of a system of rigid bars, moveable freely round their jointed extremities, is in stable equilibrium under the action of a system of forces proportional to, and bisecting perpendicularly, its several sides. Show that its several vertices lie in a circle.

(22) If a system of thin jointed bars, in which external forces act on the bars, be in equilibrium, and if the external forces acting on the bars be resolved into components acting at the joints, the

stress in each bar is the resultant of the reaction of the joint on the bar and the components at the joint of the external forces acting on the bar.

Let AB be any bar, A and B being the points of contact with the hinge-pins. Let R_1 and R_2 be the reactions of the hinge-pins on the bar, and F the external force. Resolve F into two components F_1 and F_2 acting at A and B. Then R_1, R_2, F_1, F_2 are equi-



valent to R_1, R_2, and F. Now R_1 and F_1, and R_2 and F_2 give in each case a single resultant S_1 and S_2 respectively ; and as the bar is in equilibrium under these two resultants they must be equal and opposite.

It will be noticed that S_1 and S_2 are not the reactions of the hinge-pins, but the resultant forces acting at the end points A and B which are in contact with the pins. These forces, as represented in the diagram, tend to shorten the distance AB. In actual bars a compression brings elastic forces into operation, and S_1 and S_2, having changed the length of the bar somewhat, will thus be equilibrated by the elastic stress in the bar. In dealing with rigid rods, we imagine the stress produced in the rod, though the change of length is supposed indefinitely small. The particles at A and B are thus in equilibrium under the forces S_1, S_2 respectively, and the stress in the bar. Hence that stress has the same value as S_1 or S_2,

and its directions at A and B are opposite to the directions of S_1 and S_2 at A and B respectively.

It may be mentioned that in frame-work a bar which is shortened by the forces acting on it is called a strut; one which is lengthened is called a tie.

(23) If a system of thin jointed bars in which external forces act on the bars be in equilibrium, the hinge-pins, supposed weightless, may be considered as being in equilibrium under the stresses in the bars they connect and the components at the joints of the external forces acting on the bars.

Since S_1 (fig. of Ex. 22) is the resultant of R_1 and F_1, a force equal and opposite to S_1 would equilibrate R_1 and F_1. Hence a force equal and opposite to S_1 together with F_1 would give as resultant a force equal and opposite to R_1. Now, if a hinge-pin connect two or more bars, it is in equilibrium under any external forces acting on it, together with forces equal and opposite to the reactions it exerts on the bars. Hence it may be considered as being in equilibrium under the stresses in the bars and the components at the hinge of the external forces acting on the bars, with the external forces acting on itself.

This result is of importance in engineering as enabling us to determine the stresses in framework subjected to given external forces.

(24) Three weightless bars AB, BC, CA, jointed at their extremities, are kept in equilibrium by three forces acting at the joints—P acting at A, Q at B, and R at C. Show that if S_a, S_b, S_c are the stresses in BC, CA, AB respectively, whose lengths are a, b, c respectively, and if O is the point in which P, Q, and R intersect,

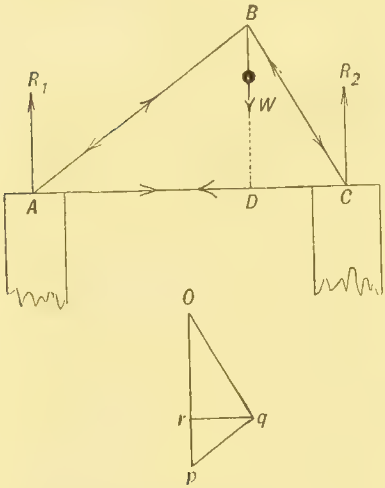
$$S_a : S_b : S_c = \frac{a\,.\,OA}{P} : \frac{b\,.\,OB}{Q} : \frac{c\,.\,OC}{R}.$$

(25) Four heavy bars AB, BC, CD, DE (weights $= w_1, w_2, w_3, w_4$ respectively) are jointed each to the next at B, C, D and to fixed points at A and E. The rod BC being horizontal, and θ being the inclination of CD to the horizon, show that the inclination of DE is

$$\tan^{-1}\left(\frac{w_2+2w_3+w_4}{w_2+w_3}\tan\theta\right).$$

(26) Three beams, AB, BC, CA are connected at their ends by smooth hinge-pins, so as to form a triangle. The ends of the beam CA rest upon pillars of equal height. The other two are in a vertical plane; and at the joint which connects them hangs a body whose weight W is so great that the weights of the beams may be neglected. The lengths of the beams being given, show how to determine the stresses in the beams and the reactions of the pillars.

Take any point O, and from it draw Op vertically downwards, making its length numerically equal to W. From O draw Oq parallel to BC, from p, pq parallel to AB, and from q, qr in a horizontal direction.

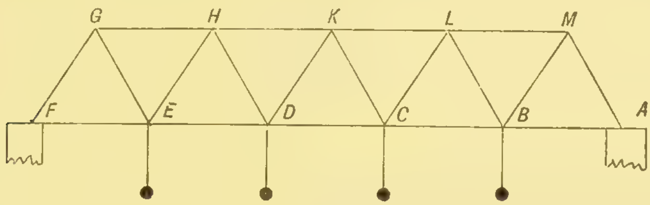


By Ex. 23 the pin at B is in equilibrium under forces whose directions are those of AB, CB, and W, or of pq, qO, and Op; and as Op represents W in magnitude it follows that pq represents the stress in AB and qO that in CB. Similarly it may be shown that qr represents the stress in CA and that pr and rO represent R_1 and R_2, the reactions of the pillars, respectively.

The diagram Opq constructed as above is called a *Force Diagram*. It may be used to solve the problem in two ways. (1) By its aid we may obtain formulæ by which the stresses may be calculated. Thus the sides of the triangle ABC being known, we may express the angles CBD and ABD and therefore the angles qOp and qpO, and therefore also the angle Oqp in terms of them. Hence also since $Oq/Op = \sin Opq/\sin Oqp$, Oq/Op may be expressed in terms of them. But if S is the stress in BC, $S/W = Oq/Op$. Hence the stress in BC may be expressed in terms of W and the lengths of the beams. And expressions for the other stresses may be obtained in the same way. (2) The lengths of the beams being given, exact values of the angles ABD, CBD may be obtained, and the force diagram may be carefully drawn to scale. Then Op having been drawn with a length numerically equal to W, a careful measurement of the lengths of

Op, pq, and qr determines the stresses. The stresses thus determined are said to be determined *graphically;* and in complicated framework the labour of calculation is much reduced by the graphic method.

(27) A Warren girder consists of 19 rods AB, BC, etc., of equal length, jointed together as in the diagram. Bodies of equal weight, and so heavy that the weights of the rods may be neglected, are



hung at the joints, B, C, D, E, and the girder is supported on piers of equal height at A and F. Show that there is no stress in KC or KD.

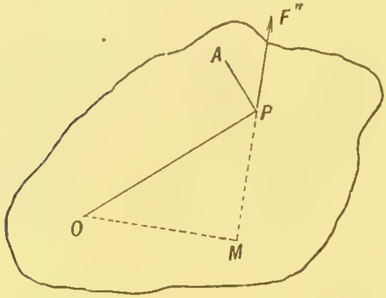
510. *Conditions of Equilibrium in terms of Work Done.*—If a rigid body is in equilibrium (*i.e.*, molar equilibrium) the algebraic sum of the amounts of work done by the external forces during any indefinitely small displacement consistent with rigidity is equal to zero.

Any such displacement may (245) be resolved into a translation, and a rotation about the direction of translation. Any point P of the body therefore will undergo a linear displacement compounded of one δ' in the direction of the translation, and another δ in a plane perpendicular to that direction, and due to the rotation about the direction of the translation. Resolve the force F, acting at this point, into two components, one F' in the direction of translation, the other F'' in the perpendicular plane.

Since the body is in equilibrium, $\Sigma F' = 0$. Multiplying by δ'. which is the same for all points of the body, we have

$$\delta' \Sigma F' = \Sigma F' \delta' = 0\,;$$

i.e., the algebraic sum of the amounts of work done by the components of the external forces in the direction of the translation is zero.



Let the plane of the diagram be the plane perpendicular to the axis of the rotation, the intersection of the two being at O. Let PA perpendicular to OP be the small linear displacement δ of P due to the rotation ω about O. From O draw OM (length $=p$) at right angles to F'', the component of F in the plane of the diagram. Since there is equilibrium, $\Sigma F''p=0$. If θ is the inclination of δ to F'',

$$p=OP\cos\theta=\frac{\delta}{\omega}\cos\theta=\frac{\delta''}{\omega},$$

if δ'' denote the component of δ in the direction of F''. Hence

$$\Sigma F''\frac{\delta''}{\omega}=\frac{1}{\omega}\Sigma F''\delta''=0.$$

Hence also $\Sigma F''\delta''=0$, *i.e.*, the work done by the components of the external forces perpendicular to the direction of the translation is zero.

If d is the component of the resultant linear displacement of P in the direction of F, Fd is the work done by F during the displacement. Hence (342)

$$Fd=F'\delta'+F''\delta''$$

and $$\Sigma Fd=\Sigma F'\delta'+\Sigma F''\delta''=0.$$

And ΣFd is the algebraic sum of the amounts of work done by the external forces during the indefinitely small displacement selected.

511. Conversely, if during any indefinitely small displacement of a rigid body, consistent with its rigidity,

the algebraic sum of the amounts of work done by the external forces be zero, the body will be in equilibrium (*i.e.*, molar equilibrium).

For it may be shown by the steps of 510 in the reverse order that

$$\Sigma Fd = \Sigma F'\delta' + \Sigma F''\delta'' = \delta'\Sigma F' + \omega\Sigma F''p = 0.$$

Now δ' and ω are arbitrary and unrelated, the displacement being any displacement whatever. Hence $\Sigma F' = 0$, and $\Sigma F''p = 0$, *i.e.*, the body is in equilibrium.

512. Hence it is a necessary and sufficient condition of the equilibrium (molar) of a rigid body that the algebraic sum of the amounts of work done by the external forces during any indefinitely small displacement consistent with rigidity, be equal to zero.

513. It follows from 449 that the necessary and sufficient condition of the molecular equilibrium (444) of a rigid body, which is obviously consistent with translation but not with rotation of the body, is that the algebraic sum of the amounts of work done by all forces, external and internal, during any indefinitely small displacement, be equal to zero. If the displacement be one consistent with the rigidity of the body, the internal forces (499) do no work. Hence the conditions of molar and of molecular equilibrium for a rigid body are identical. Thus the same conditions must be fulfilled, that a rigid body acted upon by external forces may spin without angular acceleration, as that it may move without angular velocity about an axis fixed in itself.

514. In the case of a system of rigid bodies rigidly connected [508 (1)], the necessary and sufficient condition of equilibrium is obviously that expressed in 512, the external forces involved being these external to the system.

515. In the case of a system of rigid bodies, not rigidly connected [508 (2)], since the necessary and sufficient condition of the equilibrium of each body is expressed by the equation $\Sigma Fd=0$ (510–512), the forces involved being forces external to the body, that of the equilibrium of the system is expressed by the same equation, the forces appearing in it being both those external to the system and such internal forces as stresses in strings and rods and reactions of surfaces.

If these internal forces do no work during the small displacement to which the equation applies, as will be the case if they are tensions in inextensible strings, stresses in rigid rods, or reactions of smooth surfaces, the equation $\Sigma Fd=0$ involves only forces external to the system.

516. If we wish to determine one of the internal forces of a system of rigid bodies connected by rigid rods or inextensible strings, we may imagine a small displacement in which the parts, between which the required force acts, so move that the required force does work, in which case the equation $\Sigma Fd=0$ involves the external forces and the required internal force.

517. *Examples.*

(1) A beam (weight $=W$, length $=l$, distance of centre of mass from lower end $=a$) rests with one end on a smooth horizontal plane and the other against a smooth vertical wall, in a vertical plane normal to the wall, and is prevented from sliding by a force F acting at the lower end of the beam towards the wall. Find (a) its inclination θ to the horizon, and (b) the reaction R of the vertical plane.

(a) Let the beam undergo a small displacement by which the inclination is changed from θ to $\theta+\theta'$, where θ' is small, the ends remaining in contact with the horizontal plane and vertical wall. Then the lower end moves towards the wall through a distance

$$l\cos\theta - l\cos(\theta+\theta'),$$

which, since θ' is small, is equal to $l\theta' \sin\theta$. Also the centre of mass falls through a distance

$$a \sin\theta - a\sin(\theta+\theta'),$$

which, since θ' is small, is equal to $-a\theta' \cos\theta$. No work is done by or against the reactions of the horizontal plane and vertical wall. Hence, for equilibrium,

$$Fl\theta' \sin\theta - Wa\theta' \cos\theta = 0,$$

and
$$\theta = \tan^{-1}\frac{Wa}{Fl}.$$

(*b*) Let the beam undergo a small translation in a horizontal direction, the reactions being supposed to continue during the displacement. Then the only forces by or against which work is done are F and the required reaction, and their points of application move through equal distances. Hence, if d is the translation, $Rd - Fd = 0$, and $R = F$.

(2) A body A, of weight W, is supported by a body B, of weight w, by means of the system of smooth weightless pulleys of 509, Ex. 2. Find the relation of w to W.

The only forces of the system by or against which work is done during a displacement are w and W. When A rises through a distance s, B falls through a distance $2^n s$, where n is the number of moveable pulleys. Hence

$$w2^n s - Ws = 0,$$

and
$$w/W = 1/2^n.$$

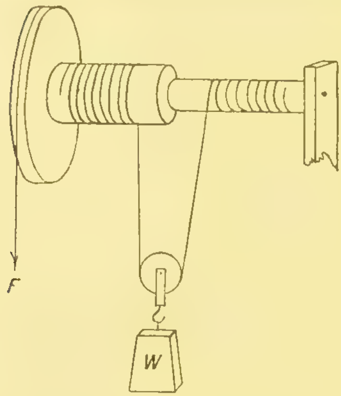
(3) Obtain the results of 509, Ex. 1 and 3, by the above method.

(4) A Wheel and Axle is used to raise a bucket from a well. The radius of the wheel is 15 in., and while it makes seven revolutions the bucket, which weighs 30 lbs., rises $5\frac{1}{2}$ feet. Find the smallest force with which the wheel can be turned.

Ans. The weight of 3 lbs.

(5) Find the mechanical advantage of the *Differential Wheel and Axle*. [In the Wheel and Axle the smaller the radius of the axle with a given radius of wheel, the less the force required to support a body of given weight hanging by a cord wrapped round the axle (254, Ex. 5 and 507, Ex. 10). To increase the mechanical advantage

of the machine without weakening the axle unduly, the cord hanging from the axle is passed round a pulley supporting the body, and so wrapped round a prolongation of the axle of smaller radius that, when it unwinds from the thicker portion of the axle, it will wind on the thinner portion. This machine is called the differential wheel and axle.]



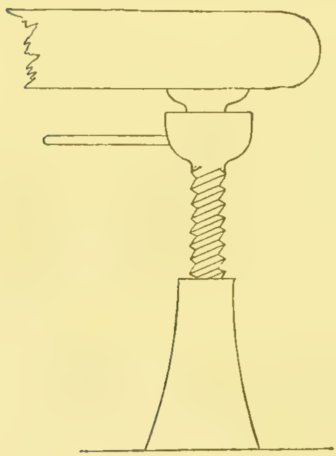
When the place of application of the force F moves down a distance s, the wheel turns through an angle s/R radians (R=radius of wheel). Hence, if r and r' are the radii of the larger and smaller portions of the axle, lengths rs/R and $r's/R$ of cord are wound on the larger portion of the axle and off the smaller portion respectively. The pulley therefore rises through a distance $s(r-r')/2R$. Hence

$$Fs-\frac{Ws}{2R}(r-r')=0,$$

and

$$\frac{W}{F}=\frac{2R}{r-r'}.$$

(6) A heavy beam presses upon the top of a smooth jack-screw with a force F. The distance in the direction of the axis of the screw between successive windings of the thread is d. Find the force P which must be applied at the end of a handle, of length l, perpendicularly to its length, to maintain equilibrium.



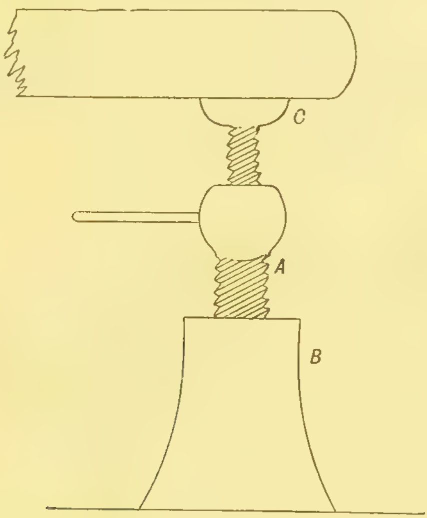
For one turn of the handle the beam would be raised a distance d, and P's point of application would move in P's direction a distance equal to $2\pi l$. Hence

$$2\pi lP-Fd=0,$$

and

$$P=\frac{Fd}{2\pi l}.$$

(7) Find the mechanical advantage of the *Differential Screw.* [The mechanical advantage of the screw (Ex. 6) increases as the distance between successive windings of the same thread diminishes. It is therefore limited by the necessity of giving the thread sufficient strength. To increase the mechanical advantage without undue diminution of strength, a combination of two screws is employed as shown in the diagram. A is a screw working in a nut cut in the



block B. Between A and the body to which force is to be applied the screw C intervenes. C works in a nut in the interior of A, its upper end being fixed so that it cannot rotate. When A advances by the amount corresponding to one turn, viz., the distance between successive windings of A's thread, C screws into A to a length equal to the distance between successive windings of C's thread, and thus C advances by an amount equal to the difference of these distances. Such an arrangement is called a *differential screw.*

Ans. $2\pi l/d$, where l is the length of the arm or handle, and d is the difference of the distances between successive windings of the threads of the respective screws.

(8) Show that the *efficiency of a machine, i.e.,* the ratio of the useful work done by it when it is moving uniformly (and therefore

is in equilibrium) to the whole amount of work done, is equal to the ratio of the force which would drive the machine against the force against which useful work is to be done, were there no friction or other forms of non-conservative force, to the force which is actually required to drive it.

If F is the force actually applied to the machine and s the displacement of the point of application in its direction, W the useful work done by the machine, and w the work done against friction and other such resistances, we have $Fs = W + w$. If F' is the force which would do the same useful work, if the friction and other resistances did not act, then $F's = W$. Hence the efficiency

$$\frac{W}{W+w} = \frac{F'}{F}.$$

(9) Find the efficiency of the rough lever of 507, Ex. 32.—Let Q be the force against which the useful work is done and P the force applied to the lever. Then (507, Ex. 32)

$$Pp = Qq + r \sin \epsilon \sqrt{P^2 + Q^2 + 2PQ \cos \theta}.$$

Let P' be the value P would have were the lever smooth. Then

$$P'p = Qq.$$

Hence the efficiency

$$E = \frac{P'p}{Pp} = \frac{Qq}{Pp}.$$

If, in the above equation, we substitute for Q its value $EPp\, q$, we obtain

$$pq(1 - E) = r \sin \epsilon \sqrt{p^2E^2 + 2pq\, E \cos \theta + q^2},$$

an equation which determines the value of E.

(10) Determine the mechanical advantage of a rough screw.

Let F be the force against which work is done, and P the force by which work is done on the handle of the screw. Let R be the normal component of the reaction of any little element of the thread. Then μR is the frictional component. In a rotation of the screw through a small angle θ, if i is the inclination of the thread to a right section of the cylinder, r the radius of the cylinder, and l the length of the arm, the work done against F is $Fr\theta \tan i$, that

done against friction is $\Sigma\mu Rr\theta \sec i$, that done by P is $Pl\theta$. Hence for equilibrium

$$Pl\theta - Fr\theta \tan i - \Sigma\mu Rr\theta \sec i = 0,$$

and
$$Pl - Fr \tan i - \mu r \sec i \,.\, \Sigma R = 0.$$

Now the equilibrium of the screw also requires that the sum of the component forces in the direction of the axis should be zero. Hence

$$F - \Sigma R \cos i + \Sigma\mu R \sin i = 0,$$

and
$$F = (\cos i - \mu \sin i)\Sigma R.$$

Substituting this value of ΣR in the former equation, we obtain

$$Pl = Fr \tan i + \mu r \sec i \frac{F}{\cos i - \mu \sin i}$$

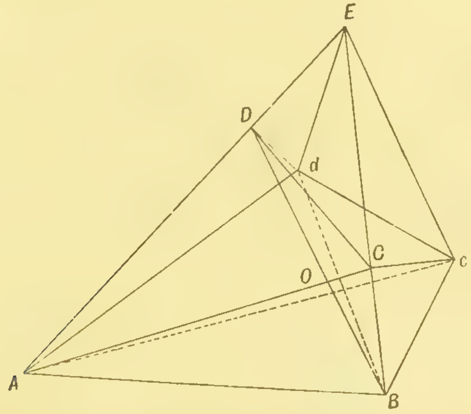
$$= Fr \tan (i + \epsilon),$$

where ϵ is the angle of repose. Hence

$$\frac{F}{P} = \frac{l}{r \tan (i + \epsilon)}.$$

(11) Show that the efficiency of a rough screw is $\tan i / \tan (i + \epsilon)$, i and ϵ having the same meanings as in Ex. 10.

(12) Four rigid weightless bars, jointed at their extremities so as to form a quadrilateral $ABCD$ in one plane, and having the opposite vertices connected by tense strings AC, BD, are in equilibrium. Compare the tensions in the strings.



The vertices A, B, C, D may (509, Ex. 23) be considered to be in equilibrium under the tensions in the strings and the stresses in

the bars. As we wish merely to compare the tensions in the strings we choose a small displacement which will involve variation in length of the strings only. Then the tensions will be the only forces appearing in the equation. Let $ABCD$ therefore undergo a small displacement of that kind, taking the form $ABcd$. The displacements of D and C will be small arcs Dd, Cc of circles about A and B as centres respectively. The elongations of the strings BD and AC will be the projections on their directions of Dd and Cc respectively. Hence the elongation of AC is $Cc \cos CcA$, which, since BCc is a right angle and Ac ultimately coincides with AC, is equal to $Cc \sin ACB$. Similarly that of BD is $-Dd \sin BDA$. Hence, if T is the tension in AC, T' that in BD, we have (515)

$$T \,.\, Cc \,.\, \sin ACB - T' \,.\, Dd \,.\, \sin BDA = 0,$$

and $$\frac{T}{T'} = \frac{Dd \,.\, \sin BDA}{Cc \,.\, \sin ACB}.$$

Now (233 and 254, Ex. 8) the instantaneous centre of the displacement of CD is the point E in which AD and BC intersect. Hence the angles DEd and CEc are equal; and therefore

$$\frac{Dd}{Cc} = \frac{DE}{CE}.$$

Also $$DE = BD \,.\, \frac{\sin DBE}{\sin BED},$$

and $$CE = AC \,.\, \frac{\sin CAE}{\sin AEC}.$$

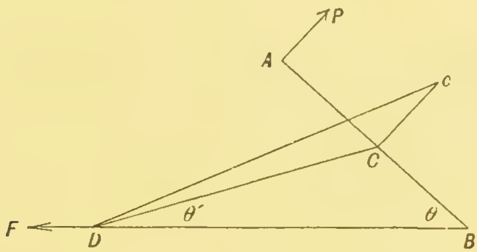
Hence $$\frac{T}{T'} = \frac{BD \,.\, \sin BDA \,.\, \sin DBE}{AC \,.\, \sin CAE \,.\, \sin ACB}$$

$$= \frac{BD}{AC} \cdot \frac{OA}{OD} \cdot \frac{OC}{OB},$$

O being the point of intersection of AC and BD.

(13) The toggle-joint consists of two bars AB and CD of which AB is moveable about a fixed joint at B, and CD is jointed to AB at C while its end D is constrained to move in the line BD. Find the relation of the force F acting on D in the direction BD, to the force P acting at A perpendicularly to AB, when there is equilibrium.

Imagine a small displacement of the system whereby, while D remains fixed, A moves through a small arc (length $=s$) of a circle



about B as centre. Then C moves through Cc, whose length is $s \,.\, CB/AB$. The elongation of the bar CD is thus

$$Cc \cos DcC = \frac{CB}{AB} s \sin(\theta+\theta'),$$

if the angles CBD and CDB are θ and θ' respectively, since the inclination of cD to cC is ultimately equal to that of CD to cC produced. Hence if S is the stress in CD

$$Ps - S\frac{CB}{AB} s \sin(\theta+\theta') = 0,$$

and
$$S = P\frac{AB}{CB\sin(\theta+\theta')}.$$

Now D is in equilibrium under S, F and the forces by which it is constrained to remain in the line BD; and if we neglect friction, these forces are perpendicular to BD. Hence $F = S\cos\theta'$, and

$$F = P \,.\, \frac{AB}{CB} \,.\, \frac{\cos\theta'}{\sin(\theta+\theta')}.$$

Hence if both θ and θ' diminish to zero, F becomes infinitely great.

The reader should solve some of the examples of 507 and 509 by this method.

CHAPTER VII.

DYNAMICS OF ELASTIC SOLIDS AND FLUIDS.

518. *Statics of Deformable Bodies.*—We have seen (256 and 268) that the motion of non-rigid bodies may be compounded of translation, rotation, and strain. In studying the effect of the exertion of force on such bodies, we in the first instance restrict ourselves to the consideration of its effect in producing strain. In other words, we consider the equilibrium of strained bodies, determining the forces necessary to maintain equilibrium when they are strained in a given manner, and the strains which will be maintained in them by given forces.

In discussing the effect of forces in producing change of the linear and angular momentum of bodies (414 and 428), we found that the internal forces might be neglected. In determining their effect in producing strain, however, both internal and external forces must be taken into account.

519. *Stresses.*—Across any surface which we may imagine as drawn in the interior of a body, innumerable forces act, the particles on the one side attracting or repelling those on the other, and the latter reacting on the former. We may regard all these forces as being a single force whose place of application is not a point but the given surface; and when so considered we call the

force a stress. Across any surface of a body then, or between the two portions into which it divides the body, we have in general a stress acting.

When the stress across a surface is one which opposes the separation of the portions of the body on opposite sides of the surface, the stress is called a pull, a tension, a traction, or a negative pressure. When it is one which opposes the approximation of these portions, it is called a push, a thrust, or a pressure.

We speak of a stress as acting *across* a surface when we wish to draw attention to its acting in opposite directions on the two portions of the body on opposite sides of the surface. When we wish to restrict attention to its action on one of these portions, we speak of it as acting *on* the bounding surface of that portion.

520. The forces acting between the particles on opposite sides of any surface in a body may have any directions and magnitudes. In general, therefore, the stress across a surface cannot be said to have any one direction or magnitude. In the important case of a continuous stress, however, the case, *i.e.*, in which the resultant forces acting on particles indefinitely near one another have indefinitely nearly the same magnitude and direction, if an indefinitely small part of the surface be taken, the stress across it may be considered as acting at a point, and as having both a definite direction and a definite magnitude.

521. *Integral Stress over a Surface.*—If any given surface be divided into an indefinitely large number of indefinitely small portions, the sum of the forces on these small portions may be called the integral stress over the surface. If the surface is finite, it is obviously a quantity having magnitude, but in general not direction.

The mean stress over a surface is the quotient of the integral stress over the surface by its area. For

a finite surface it is also a quantity in general without direction.

The stress at a point across a given surface through the point has a magnitude which is the limiting value of the mean stress over a portion of the given surface containing the point, when the area of that portion is made indefinitely small. By 520 it will have a definite direction in cases of continuous stress.

The magnitude of the mean stress over a surface or of the stress at a point is usually spoken of as its intensity.

The stress at a point is in general different for different points of any given surface, both as to magnitude and direction. If the stress has at all points the same magnitude and direction it is said to be uniform over the surface.

The stress at any point is in general different both as to magnitude and direction for different surfaces through the point.

522. *Homogeneous Stress.*—If the stresses at all points of a body across parallel surfaces through them are the same, the stress is said to be homogeneous throughout the body. If not, the stress is said to be heterogeneous.

Heterogeneous stresses are in general continuous, *i.e.*, the stresses across parallel surfaces at points indefinitely near one another are indefinitely nearly the same. It is obvious that if a body be subjected to a continuous heterogeneous stress, the stress may be taken to be homogeneous throughout indefinitely small portions of it.

523. *Measurement of Stress*—The intensity of a mean stress over a surface, or of the stress at a point of a surface, being the quotient of a force by the area of a surface, the derived unit of stress will be unit force per

unit of area, *e.g.*, one poundal per sq. foot, one dyne per sq. centimetre, one pound-weight per sq. inch (usually expressed as one pound per sq. in.), etc.

The dimensions of the unit of stress are thus, if $[F]$ and $[S]$ are the magnitudes of the units of force and area respectively, $[F][S]^{-1}$, and therefore by 303 $[M][L]^{-1}[T]^{-2}$. The reduction of the numerical values of stresses from one to another system of units is made after the same manner as in the case of speed, rate of change of speed, etc. (45-50, 56-59).

524. *Examples.*

(1) Show that a stress of 20 poundals per sq. foot is equivalent to one of ·2975 dynes per sq. centimetre approximately.

(2) One pound-weight per sq. inch is equivalent to $6{\cdot}9 \times 10^4$ dynes per sq. cm. nearly.

(3) Reduce 40 dynes per sq. cm. to kilogrammes per sq. dcm.
Ans. $4{\cdot}08 \times 10^{-3}$ nearly.

(4) The unit of stress of a derived system being the poundal per sq. in., the unit of mass a mass of 2,000 lbs., and the unit of time a minute, find the unit of length.
Ans. 0·00386 ft. nearly.

525. *Resultant of Stress on a Surface.*—It is frequently convenient to imagine the portion of any non-rigid body under consideration to become rigid, and to treat it as though acted upon by the forces, acting at points, which in that case would produce in it the same effect as the stresses on its bounding surfaces. This course is admissible, because, if a portion of a deformable body be in equilibrium under stresses acting on it over its bounding surfaces, it will still be in equilibrium, though it become rigid; and if it become rigid, it will still remain in equilibrium, though one or more of the stresses acting on its bounding surfaces be replaced by equivalent forces acting at isolated points.

526. It is therefore important to determine the single force or the simplest system of forces to which a stress on a given surface may be equivalent, or, as it may be called, the resultant of the stress. In general such a stress will not have a single force as resultant (476, 477). But in the special case in which the stresses at all points of a surface have the same direction, a single resultant may be found (471).

To find it, divide the whole surface of area S into a large number of small portions of areas s_1, s_2, etc. Then $S=\Sigma s$. If p_1, p_2, etc., are the values of the mean stress over the areas s_1, s_2, etc. (when these areas are indefinitely diminished, the mean stresses become stresses at a point), the integral stresses over these areas will be $p_1 s_1$, $p_2 s_2$, etc.; and as these stresses are parallel, we have (465, 470-1), if P is the magnitude of the resultant stress,

$$P=p_1 s_1+p_2 s_2+\text{etc.}=\Sigma ps.$$

The direction of P will be the common direction of the stresses p_1, p_2, etc., or, in other words, the direction of the stress at any point of the surface.

The magnitude of the integral stress over a surface, when the stresses at its points have different directions, is obviously equal to that of the resultant stress over the same surface when the stresses at its points have the same intensities and have also a common direction.

527. *Examples.*

(1) Find the resultant stress over a surface of area S, the stress at all points of the surface having the uniform intensity p, and a uniform direction.

Ans. pS. (For $\Sigma ps=p\Sigma s=pS$.)

(2) Find the integral stress over a surface of area S, and consisting of indefinitely small portions s_1, s_2, etc., whose distances from a given plane are h_1, h_2, etc., respectively, the stress at any point of

the surface being proportional to the distance of the point from the given plane.

If p_1, p_2, etc., are the intensities of the stress on s_1, s_2, etc., respectively, we have $p_1 = kh_1$, $p_2 = kh_2$, etc., where k is a constant. Hence the forces acting across s_1, s_2, etc., are $kh_1 s_1$, $kh_2 s_2$, etc. Hence the integral stress is

$$\Sigma khs = k\Sigma hs = kS\Sigma hs / \Sigma s,$$

for $S = \Sigma s$. Now $\Sigma hs/\Sigma s$ is obviously (400) the distance from the given plane of the centre of mass of a uniform thin material lamina of the same form and area as the given surface, and of surface density unity (304), or, as it is called for shortness, the *centre of mass of the surface.* Hence the integral stress is equal to the product of the constant k, into the area of the surface, into the distance of its centre of mass from the given plane.

(3) Find the resultant of a normal stress on a plane surface of rectangular form (sides $= a$ and b), the stress at any point being proportional to its distance from a given plane parallel to the sides of length a and inclined to the sides of length b at the angle θ, and that side of length a which is nearest the given plane being at a distance h from it. (Use result of Ex. 2.)

Ans. $kab(b \sin \theta + 2h)/2$.

(4) Find the integral stress over a spherical surface of radius r, the stress at any point being proportional to its distance from the tangent plane at the highest point of the sphere and the stress at a point at unit distance being k.

Ans. $4\pi kr^3$.

(5) Find the integral stress over the curved surface of a right cone of height h and semi-vertical angle θ, the stress at any point of it being numerically equal to p times the distance of the point from the base.

Ans. $\pi ph^3 \sin \theta / 3 \cos^2 \theta$.

528. *Centre of Stress.*—If a single force can be found which is equivalent to a given stress on a given surface, its point of application is called the centre of the stress.

In the case in which the stresses at all points of the surface are parallel, the centre of the stress is the centre of the system of parallel forces of which the stress may be regarded as consisting. Hence, in this special case, if the surface consist of small portions of areas s_1, s_2, etc., at which the intensities of the stress are p_1, p_2, etc., if the distances of s_1, s_2, etc., from any plane are h_1, h_2, etc., and if the distance from the same plane of the point of application of the resultant (Σps, by 526) is H, we have by 472

$$H = \Sigma psh / \Sigma ps.$$

If the distances of the centre of stress be determined from any three intersecting planes, its position is completely specified.

529. *Examples.*

(1) Show that the centre of stress for any plane surface subjected to a uniform stress is the centre of mass of the surface.

$$H = \Sigma psh/\Sigma ps = p\Sigma sh/p\Sigma s = \Sigma sh/\Sigma s. \quad \text{[See 527, Ex. 2.]}$$

(2) Find the centre of stress for a plane triangle, the stresses at all points being uniform in direction and varying as the distances of the points from a plane through one of the sides.

If the triangle be divided into narrow strips of equal width parallel to this side, the stress will be uniform over each strip. Hence the centre of stress for each strip is its middle point, and that of the whole triangle is on the line drawn from the middle point of the above-mentioned side to the opposite angle. The resultant stresses on strips equidistant from the middle point of this line may easily be shown to be equal. Hence the middle point of this line is the centre of stress for the triangle.

(3) Find the centre of stress on a parallelogram $ABCD$, the stress at all points being uniform in direction and varying as their distance from a plane through AB.

If the parallelogram be divided into narrow strips of equal width parallel to AB, the resultant stress on each will act at its middle point and be proportional to its distance from the given plane, and

therefore to its distance from AB. Hence the resultant stresses on the strips are proportional to the lengths of the portions of the strips intercepted between straight lines drawn from C and D to E, the middle point of AB; and hence the centre of stress of the parallelogram coincides with the centre of mass of the triangle ECD.

(4) Find the centre of stress for any plane surface, the stresses at its various points being parallel and proportional to their distances from any given plane.

With the symbols of 528 we have $p_1 = kh_1$, $p_2 = kh_2$, etc. Hence

$$\Sigma psh/\Sigma ps = \Sigma sh^2/\Sigma sh.$$

The determination of the value of $\Sigma sh^2/\Sigma sh$ in special cases requires in general the application of the Integral Calculus.

(5) Find the centre of stress on a triangular plane ABC, the stresses at all points being uniform in direction and proportional to the distances of the points from a plane through C parallel to AB.

Let the triangle be divided into n narrow strips of equal width parallel to AB. These may be treated as rectangles if n be very great. If AB have the length a, and if b be the distance of C from it, the areas of these rectangles in the order in which they occur from C towards AB are ab/n^2, $2ab/n^2$, $3ab/n^2$, etc. As they are very narrow the distances of their centres of mass from the given plane, if h is the distance of AB from it, may be taken to be h/n, $2h/n$, $3h/n$, etc. Hence the distance of the centre of stress from the given plane is (Ex. 4)

$$\frac{abh^2/n^4 + 2^3abh^2/n^4 + \text{etc.} + n^3abh^2/n^4}{abh/n^3 + 2^2abh/n^3 + \text{etc.} + n^2abh/n^3} = h \cdot \frac{(1^3 + 2^3 + \text{etc.} + n^3)/n^4}{(1^2 + 2^2 + \text{etc.} + n^2)/n^3}$$
$$= 3h/4,$$

since n is indefinitely great. And it is obviously in the line joining C with the middle point of AB.

(6) Find the distance from a given plane of the centre of stress on a triangle ABC, the point A being in the given plane and the points B and C at distances h_1 and h_2 from it, the stress at any point being normal and proportional to the distance of the point from the given plane. [Let BC meet the given plane in D. Then the resultant stresses on ACD and ABD may be determined in terms of the length of AD and the inclination of the plane of ABC

to the given plane, and their centres of stress may be determined by Ex. 2. Then the resultant stresses and the centres of stress on the whole ABD and the part ACD being known, the centre of stress of the part ABC may be readily determined.]

Ans. $(h_1^2+h_1h_2+h_2^2)/2(h_1+h_2)$.

(7) Find the centre of stress on a parallelogram $ABCD$, the stress at any point being normal and proportional to its distance from a given plane which is parallel to the sides AB and CD, and distant h_1 and h_2 from them respectively.

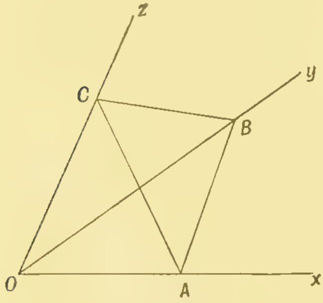
Ans. $2(h_1^2+h_1h_2+h_2^2)/3(h_1+h_2)$.

530. *Resolution of Stress.*—A stress, being in general oblique to the surface across which it acts, may be resolved into tangential and normal components. For each of the forces acting at points, of which it may be considered to consist, may be so resolved.

A stress which is normal to the surface across which it acts is often called a *longitudinal stress.* One which has the inclination zero is called a *tangential or shearing stress.*

531. *Specification of Stress.*—The magnitudes and directions of the stresses at a point across any three plane surfaces through the point being given, the stress across any other plane through the point can be determined.

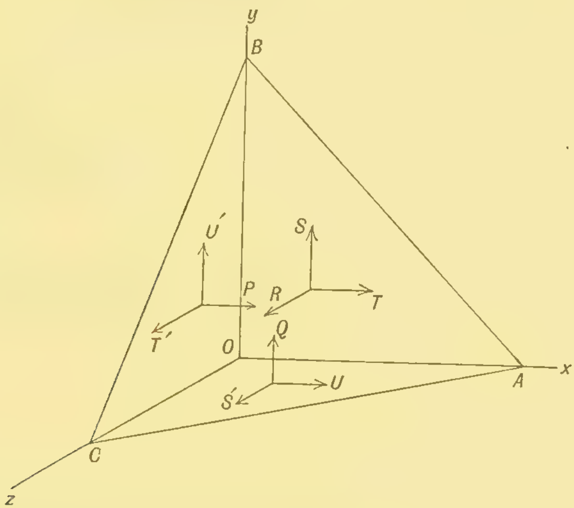
First, let the stress throughout the body be homogeneous, and let there be no external forces. Let O be the given point, and Ox, Oy, and Oz the intersections of the three planes through O; and let any fourth plane intersect these planes in AB, BC, CA. Then the tetrahedron $OABC$ being in equilibrium under the resultant stresses on its four faces, and those on the three faces OAB, OBC, OCA being known, the magnitude and direction of that on ABC may



be determined by 500; and the area ABC being known, the stress at any point of ABC, and consequently the stress at O, across a plane parallel to ABC becomes known.

532. It is usually convenient to take rectangular planes as planes of reference.

Let $OABC$ be a tetrahedron whose faces OAB, OBC, OCA are at right angles to one another; and let the normal to the plane ABC have the direction cosines l, m, n relative to the x, y, z axes respectively. Let the stress



at O across OAB (the xy plane) have components T, S, R in the directions of Ox, Oy, Oz respectively, that across OBC (the yz plane) components P, U', T', and that across OAC (the xz plane) components U, Q, S', in the same directions respectively. Also let F_x, F_y, F_z be the components in these directions of the stress F at O across a plane parallel to ABC, and therefore across ABC. Then the tetrahedron is in equilibrium under forces equal to the products of these various stresses into the areas of the faces across which they act, and acting (529, Ex. 1) at the centres of mass of the faces. Hence (500)

$$F_x \,.\, ABC = P \,.\, OBC + U \,.\, OAC + T \,.\, OAB,$$

ABC, OBC, etc., standing for the areas of the faces. Now OBC, OAC, and OAB are the projections of ABC on the yz, xz, and xy planes respectively. Hence (see 173)

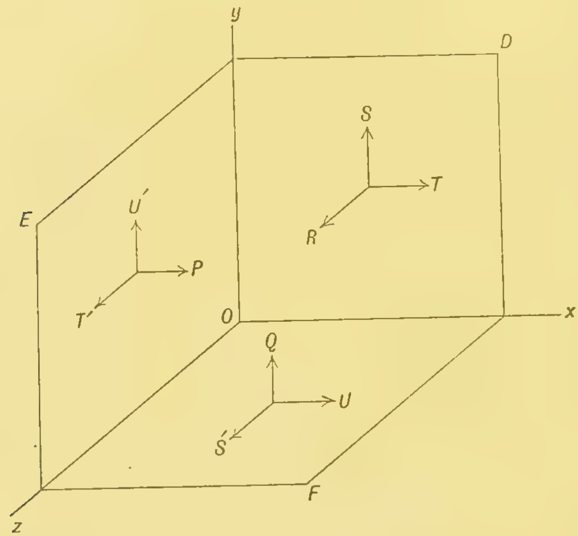
$$OBC = ABC \,.\, l\,;\quad OAC = ABC \,.\, m\,;\quad \text{and } OAB = ABC \,.\, n.$$

Hence $$F_x = Pl + Um + Tn.$$

Similarly $$F_y = U'l + Qm + Sn,$$

and $$F_z = T'l + S'm + Rn.$$

533. It is also necessary for equilibrium (500) that the sum of the moments of the acting forces about Ox, Oy, Oz should be equal to zero. The relations between the components of the stresses, which are obtained by applying this condition, however, may be more easily obtained by considering the equilibrium of a cube of which Ox, Oy, Oz are adjacent edges. Let OD, OE, OF be three faces of such a cube. The component stresses at all points of



these faces are the same as at all points of the corresponding faces of the tetrahedron; and the component stresses at all points of the faces opposite to OD, OE, OF are equal and opposite to those on OD, OE, OF respectively. Let the component stress equal and opposite to P on the

face opposite to OE be called p, and let the stresses similarly related to Q, R, etc., be called q, r, etc. If the cube be one of unit edge, the components of the resultant stresses on its faces are P, Q, R, etc., p, q, r, etc., and the points of application of these component forces are the centres of the faces. Hence, equating to zero the sum of the moments about Ox of all the forces acting on the cube, and noting that P, T, U, p, t, u, which are parallel to Ox, and S and S' which intersect it, have no moments about it, that R and r, Q and q, T' and t', and U' and u' have equal and opposite moments about Ox, and that s and s' are equidistant from it and have moments of opposite sign about it, we obtain $s=s'$, and therefore

$$S=S'.$$

Similarly we find $$T=T',$$

and $$U=U'.$$

534. Substituting these values of S', T', U', in the expressions of 532 for F_x, F_y, F_z, we have

$$F_x=Pl+Um+Tn,$$
$$F_y=Ul+Qm+Sn,$$
$$F_z=Tl+Sm+Rn.$$

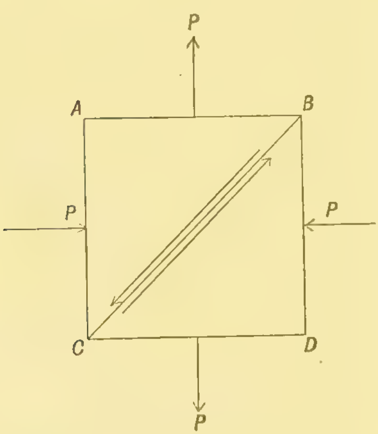
535. Hence if P, Q, R, S, T, U are known, the stress at O across any surface through O is known. The complete specification of the stress at a point requires then only these six numerical data. P, Q, and R are the component stresses at O, normal to the yz, xz, and xy planes respectively. S is the tangential or shearing stress either on the xy plane parallel to the y axis, or on the xz plane parallel to the z axis; T, that on the xy plane parallel to the x axis, or on the zy plane parallel to the z axis; U, that on the xz plane parallel to the x axis, or on the yz plane parallel to the y axis.

536. Secondly (531), if the stress is not homogeneous the same result may be obtained, provided the tetrahedron

and cube above be taken indefinitely small. For in that case the stresses at O across the planes of the faces may be taken to be the stresses at all points of the faces.

537. The above conclusions (534-5) hold also if the body is acted upon by external forces. Such forces must either be forces acting on the outer surface of the body, or forces, such as gravitational attraction, acting throughout the body. Forces acting at the outer surface of the body act only on tetrahedra or cubes having faces in the bounding surface, and they constitute the stresses on those faces. Forces acting on all parts of the body are proportional to the masses of the parts acted upon. Hence such of these forces as act on the tetrahedron or cube are proportional to its volume. The stresses on its faces are proportional to the areas of these faces. The former are therefore proportional to the cubes, and the latter to the squares of any edge. Hence, if the tetrahedron or cube be gradually diminished, the external forces diminish more rapidly than the stresses; and if it be made indefinitely small, the external forces become indefinitely small relatively to the stresses, and may therefore be neglected.

538. *Resolution of a Tangential Stress into Longitudinal Stresses.*—Let a body be subjected to a tension P in a given direction, and a pressure of the same intensity in a perpendicular direction, the state of stress being homogeneous. And let $ABDC$ be a section of a cube of unit edge, with faces normal to the directions of the tension and pressure, through the central points of the faces. Then the resultants of these stresses on the faces of the cube may be considered as acting at



the middle points of the sides of $ABDC$ and as having the magnitudes P. The triangle ACB, or rather the triangular prism of which ACB is a section, being in equilibrium under the two forces P, and the resultant stress on CB, this resultant stress must be equal and opposite to the resultant of the two forces P, on AB and AC. Now in a direction normal to CB, these forces have equal and opposite components, and in the direction of CB each has a component $P \cos 45°$. Hence the resultant stress on CB must be in the direction BC, and of the magnitude $2P \cos 45°$. Now the section of the cube through CB perpendicular to $ABDC$, which is the surface on which this stress acts, has the area $1/\cos 45°$. Hence the intensity of the stress on CB is $2P \cos^2 45°$ or P.

Hence a tension parallel to one line, and an equal pressure parallel to any line at right angles to it, are together equivalent to a shearing stress of the same value on planes cutting these directions at angles of 45°. (Compare 276.) The directions of the pressure and tension may be called the axes of the shearing stress.

539. It follows that since a stress at any point of a body may be completely specified in terms of longitudinal and shearing stresses, it may also be completely specified in terms of longitudinal stresses alone.

540. *Relation of Stress to Strain.*—In considering the determination of the strain produced in a body when subjected to given stresses, we must restrict ourselves to the simple case in which the body is homogeneous, isotropic, and perfectly elastic.

541. A body is said to be *homogeneous* provided any two equal, similar and similarly situated parts of it are not distinguishable from one another by any difference in quality. Probably no bodies perfectly fulfil this condition without limit as to the smallness of the parts.

But many bodies are so nearly homogeneous that their heterogeneity eludes observation.

542. A homogeneous body is said to be *isotropic*, when any two equal and similar portions of it, whether similarly situated or not, are not distinguishable from one another, or, in other words, when it has the same qualities in all directions. A body which exhibits differences of quality in different directions is said to be *æolotropic*.

A body may be isotropic with respect to some qualities, and æolotropic with respect to others. We have to do with isotropy only with respect to the relations of stress to strain.

543. A body is said to be *elastic*, provided (1) the application of force is required to produce a change in its shape or its bulk; and (2) a continued application of force is necessary to maintain the change, in which case it will return towards its initial shape or bulk when the applied force is removed.

A body is said to be *perfectly elastic* for a strain of a given kind, provided the same application of force is requisite to maintain the given strain as to produce it, in which case it will obviously return to its initial configuration when the stress is removed.

544. Probably no natural bodies fulfil this condition of perfect elasticity, unless in producing strains in them care be taken to keep them at constant temperature. For in all bodies the stress required to maintain a given strain is found to vary with temperature; and we know from Thermodynamics that consequently a change of configuration must be accompanied by a change of temperature.

545. In all bodies it is found that the amount by which the stress required to produce a strain exceeds that re-

quired to maintain it, is greater than the amount due merely to this change of temperature: and the difference between these amounts is found to depend upon the rapidity with which the change of configuration is produced. Thus the relative motion of the parts of a body are resisted in the same way as the relative motion of different bodies in contact; and bodies are therefore said to exhibit *molecular friction*, or as it is called *viscosity*.

Even a perfectly elastic body will not therefore appear to be perfectly elastic unless its changes of configuration are carried out with infinite slowness.

546. For most bodies, and for most kinds of strain, there are limiting values of the stress by which a strain of a given kind is produced, within which the elasticity for that kind of strain is perfect, and beyond which the elasticity is imperfect. Such limiting value of the stress is called the *limit of perfect elasticity* for that kind of strain.

547. All bodies exhibit some degree of elasticity of volume. If a body possess any degree of elasticity of shape, it is called a *solid*. If a body possess no degree of elasticity of shape, it is called a *fluid*.

548. That a body may be elastically isotropic, *i.e.*, isotropic so far as the relation of stress to strain is concerned, it must obviously satisfy two conditions:—(1) Any spherical portion of it must, if subjected to a uniform normal pressure or tension over its whole surface, undergo no deformation, the compression or dilatation produced being the same in all directions; (2) Any cubical portion of it, subjected to shearing stresses on the planes of its faces, must undergo distortion or shear; and the amount of the shear must be the same to whatever side of any face the shearing stress is parallel.

549. Hence the relation of stress to strain in a perfectly elastic homogeneous isotropic body is completely defined if we know (1) the ratio of the intensity of the stress, uniform in all directions, to the dilatation or condensation (266) which is produced by it; and (2) the ratio of the intensity of the shearing stress to the amount of the shear produced by it. The former of these ratios is called the *resistance to compression* or the *elasticity of volume*, the latter the *rigidity* or the *elasticity of figure or form.* The former may be denoted by the symbol k, the latter by the symbol n.

550. The strains produced in deformable bodies by the forces to which they may be subjected, and the relative motion of their parts under the stresses thus called into play, may be investigated either on the assumption that bodies may be regarded as consisting of discrete particles exerting forces upon one another at a distance, or on the assumption that they may be regarded as consisting of contiguous elements exerting forces on neighbouring elements only, across the surfaces of contact.

If the former fundamental assumption is made, the Laws of Motion to be employed are those given in 440, viz., the Law of Force—Newton's Second Law of Motion, and the Law of Stress, that the forces may be considered to be attractions or repulsions with magnitudes varying only with the distances between the particles. In addition, in dealing with particular classes of bodies, *e.g.*, elastic solids, assumptions must be made as to how the forces acting between the particles depend upon the distance.

If the latter fundamental assumption is adopted (and it is on this assumption that the above discussion of stresses is based), Newton's Second Law is an appropriate Law of Force, but the Law of Stress must obviously be thrown into a different form. It might readily be

given an appropriate form if it were permissible to use the terminology of the higher mathematics; but for the readers of this book it may be expressed more intelligibly by the statement that the stresses between the elements of elastic bodies are such as satisfy Newton's Third Law and the law of the Conservation of Energy. In addition assumptions must be made with respect to the laws of the stresses called into play in the case of particular classes of bodies.

551. *Statics of Elastic Solids.—Hooke's Law* gives us the necessary experimental basis for the study of the strains of elastic solids. Hooke expressed the law as follows: "*Ut tensio sic vis;* That is, The Power of any Spring is in the same proportion with the tension thereof: That is, if one power stretch or bend it one space, two will bend it two, and three will bend it three, and so forward." In modern phraseology it takes the following form: *Strain is proportional to stress.* This law has been subjected to the most minutely accurate experimental tests, and the simple proportionality of stress to strain is found to hold in the case of all solids for sufficiently small strains, and in the case of metals and hard solids (*i.e.*, solids in which the stress applied, if maintained, does not produce a continually increasing strain) for all strains within the limits of perfect elasticity.

The strains, by the investigation of which Hooke's law has been established, viz., the stretching of wires by appended weights, the compression of rods, the flexure of beams, the extension of spiral springs, the torsion of wires, etc., are all more or less complex strains, involving in most cases both change of volume and change of form. The constancy of the ratio of stress to strain, within the limits of perfect elasticity, in strains involving both change of form and change of volume, warrants us in assuming that within the same limits the elasticity of figure and the elasticity of volume must be constant also.

552. *Moduluses of Elasticity.*—A modulus of elasticity is the ratio of the intensity of a stress to the magnitude of the strain which it produces. Thus the elasticity of figure (n) and the elasticity of volume (k) are moduluses of elasticity. The elasticity of figure is often called therefore the *modulus of rigidity* (or of *simple rigidity*), and the elasticity of volume the *modulus of bulk elasticity.* The reciprocal of the latter is called the *compressibility* of the body.

Young's modulus, or the *modulus of simple longitudinal stress*, is the ratio of the intensity of the stress applied at the end of a wire or rod in the direction of its length to the increase or diminution which each unit of its length undergoes, the strain being one within the limits of perfect elasticity. The extension of a wire or rod by longitudinal stress involves change of both volume and form. Hence Young's modulus ought to be expressible in terms of k and n. (See 556 (4).)

A modulus of elasticity, being the ratio of a stress to a strain, has the same dimensions as a stress; for a strain is the ratio of two quantities of the same kind, two lengths, for example, or two volumes, and has therefore no dimensions. The dimensions of a modulus of elasticity are thus $[M][L]^{-1}[T]^{-2}$. The value of such a modulus expressed in any one system of units may thus readily be reduced to any other system of units. Moduluses are usually expressed in gravitational measure, in pounds (*i.e.*, pounds-weight) per square inch, *e.g.*, or in grammes (*i.e.*, grammes-weight) per square centimetre.

In the measurement of moduluses however a special unit of force is frequently employed, viz., the weight of unit of volume of the substance to which the modulus applies. The value of the modulus thus expressed is to be obtained from its value expressed as above in ordinary units of stress by dividing by the weight of unit volume of the substance, *i.e.* (304), by the product of the specific

gravity of the substance into the weight of the unit volume of water at the standard temperature. Thus, if a modulus be expressed in pounds per square inch, its value in terms of the special unit of force is obtained by dividing by the product of the specific gravity of the substance into the weight of a cubic inch of water, which in gravitational units is equal to the density of water in pounds per cubic inch. If the modulus be expressed in grammes per square centimetre, its value has to be divided only by the specific gravity of the substance, for the density of water in grammes per cubic centimetre may be taken to be unity.

The dimensions of "weight of unit volume" being $[F][V]^{-1}$ (where $[F]$ and $[V]$ are the magnitudes of the units of force and volume respectively), and therefore $[M][L]^{-2}[T]^{-2}$, those of moduluses expressed in terms of the weight of unit volume as unit of force are $[M][L]^{-1}[T]^{-2}/[M][L]^{-2}[T]^{-2}$ or $[L]$. The modulus thus expressed is therefore a length, and its value is therefore usually called the "length of the modulus." Thus the value of a modulus obtained by dividing its value in pounds per square inch by the product of the specific gravity of the substance into the density of water in pounds per cubic inch, is the length of the modulus in inches.

The term modulus is also applied to the following ratios, though they are not the ratios of stresses to strains :—

The *modulus of torsion* of a rod or wire is the ratio of the couple applied at one end (the other end being fixed) to the torsion produced per unit length of the wire.

The *modulus of flexural rigidity*, in any plane, of a rod or beam, slightly bent in that plane, is the ratio of the couple producing the curvature to the curvature thereby produced.

The dimensions of the modulus of torsion are obviously $[M][L]^3[T]^{-2}$; those of the modulus of flexural rigidity the same.

553. *Examples.*

(1) The modulus of rigidity of a piece of glass is 245×10^6 grammes per sq. cm. Express it (*a*) in kilogrammes per sq. mm.; (*b*) in absolute C.G.S. units; and (*c*) in pounds per sq. in.

Ans. (*a*) 2,450; (*b*) 240×10^9; (*c*) $3{\cdot}48 \times 10^6$.

(2) The modulus of bulk-elasticity for steel is $1{,}841 \times 10^9$ dynes per sq. cm. Show that its value in grammes per sq. cm. is $1{,}876 \times 10^6$, and in poundals per sq. ft. $1{,}237 \times 10^8$.

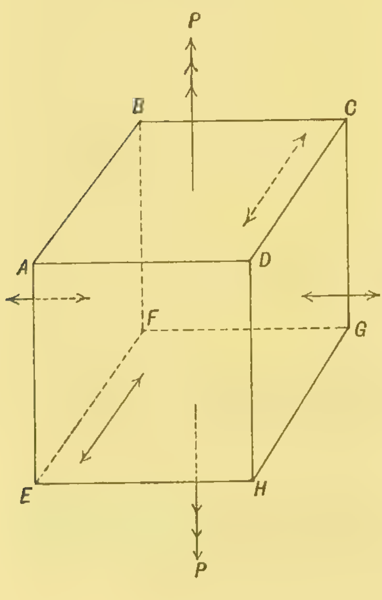
(3) Young's modulus for lead (specific gravity $= 11{\cdot}2$) being 177×10^6 grammes per sq. cm., show that the length of the modulus is $15{\cdot}7 \times 10^6$ cm.

(4) The length of Young's modulus for iron (specific gravity $= 7{\cdot}5$) being 9×10^6 feet, show that its value in grammes per sq. cm. is $2{,}057 \times 10^6$, and in pounds per sq. ft. $4{,}218 \times 10^6$. (A cubic foot of water weighs 1000 oz. approximately.)

(5) The modulus of torsion of a certain wire has the value 12×10^4 in the gravitational C.G.S. system. Find its value in the absolute ft.-lb.-sec. system.

Ans. 9·17.

554. *Strain due to Longitudinal Stress.*—As the stress at any point of a body may (539) be completely specified in terms of simple longitudinal stresses, the determination of the strain produced by any given stress requires only that we should determine the strain produced by a simple longitudinal stress.—Let AG be a unit cube of a body subjected to a simple longitudinal stress, of intensity P, normal to the faces $ABCD$ and $EFGH$. We may obviously apply to each of the other faces two equal and opposite normal stresses of the intensity $P/3$. (Each



arrow-head in the figure denotes a stress of the intensity $P/3$.) Then it is evident that the simple longitudinal stress P is equivalent to a uniform dilating tension $P/3$, together with two distorting stresses (538), each equal to $P/3$ and having one axis in the direction of the simple longitudinal stress, their other axes being at right angles to it and to one another. Hence (549) the effect of the simple longitudinal stress P will be a uniform cubical dilatation of the amount (per unit of volume) $P/3k$, together with two shears, each of the amount $P/3n$ and having one axis in the direction of P, their other axes being perpendicular to it and to one another. Each of these shears, if small, is (276) equivalent to a positive elongation equal to $P/6n$ in the direction of P and a negative elongation of the same magnitude in the direction of the other axis. Also the cubical dilatation $P/3k$ is (266) equivalent to an elongation the same in all directions and equal to $P/9k$. Hence the effect produced by P is a positive elongation in its own direction equal to

$$P/9k+P/3n \text{ or } P(3k+n)/9kn,$$

and a positive elongation equal to

$$P/9k-P/6n \text{ or } P(2n-3k)/18kn,$$

in each of two perpendicular directions at right angles to one another, and therefore in all directions at right angles to that of P.

555. *Stress required for Longitudinal Strain.*—Similarly, as any strain may (279) be specified in terms of simple longitudinal strains, the determination of the stress required to produce a given strain requires only that we should determine the stress required to produce a simple longitudinal strain.

By 277 (Ex. 1) a small simple elongation e is equivalent to a cubical dilatation e (due to elongations $e/3$ uniform in all directions), together with two shears, each of the amount $2e/3$, having the direction of the given simple

elongation as major axis or axis of positive elongation, and having as other axes lines perpendicular to the direction of the elongation and to one another. For the production of the cubical dilatation e a tension ke, uniform in all directions, is necessary. For the production of each of the shears (538) a tension in the direction of the elongation, and of the intensity $2en/3$, together with a pressure of the same intensity in a perpendicular direction are necessary, the pressures required for the two shears being perpendicular to one another. Hence the elongation e requires altogether a tension in the direction of the elongation of the intensity $(k+4n/3)e$, and tensions of the intensity $(k-2n/3)e$ in two directions perpendicular to that of the elongation and to one another, and therefore in all directions perpendicular to that of the elongation.

556. The above results are sufficient to enable us to solve a few important problems on the strains produced in elastic solids when subjected to given stresses, and on the stresses required to produce or maintain in them given strains.

Examples.

(1) A rod, bar, or wire is subjected to equal and opposite forces acting at its ends in the direction of its length. Find the ratio (called *Poisson's ratio*) of the linear contraction it undergoes laterally to the elongation produced in the direction of its length.

Ans. Obviously from 554, $(3k-2n)/2(3k+n)$.

(2) Find in Ex. (1) the diminution, per unit area of the cross section of the rod, P being the intensity of the stress applied at the ends.

Ans. $P(3k-2n)/9kn$.

(3) Show that in Ex. (1) the dilatation per unit volume is $P/3k$, P being the intensity of the stresses at the ends of the rod.

(4) Express Young's modulus in terms of the moduluses of bulk-elasticity and of rigidity.

The stress P applied at the end of a rod or wire in the direction of its length will (554) produce an elongation per unit of length of $P(3k+n)/9kn$. Hence Young's modulus, the ratio of this stress to the elongation produced, is equal to $9kn/(3k+n)$.

(5) Show that in the extension of a band of India-rubber, for which k is large in comparison with n, the area of the cross-section is diminished in nearly the same proportion as that in which the band is lengthened, and that there is therefore but little change of volume.

(6) Find (a) the stress produced at any point in a circular cylinder of length l, one end of which is fixed while the other is twisted through an angle θ, and (b) the moment of the couple which must be applied at the free end of the cylinder to maintain the torsion.

(a) By 277, Ex. (3), the cylinder is, at every point distant r from the axis, subjected to a shear whose plane is perpendicular to a plane through the point and the axis, and is parallel to the axis, whose direction is normal to the plane containing the point and the axis, and whose amount is $\theta r/l$. Hence the stress at any point is a shearing stress of the intensity $n\theta r/l$, on a plane normal to the axis and in a direction perpendicular to a plane through the axis and the given point.

(b) If the normal section at the end of the cylinder be divided into an indefinitely large number of indefinitely small portions of areas s_1, s_2, etc., distant r_1, r_2, etc., from the axis, the resultant shearing stresses on them will be $n\theta r_1 s_1/l$, $n\theta r_2 s_2/l$, etc. The moments of these resultants about the axis will be $n\theta r_1{}^2 s_1/l$, $n\theta r_2{}^2 s_2/l$, etc. Hence, if T is the moment of the couple which must be applied at the free end to maintain the given torsion,

$$T = n\theta r_1{}^2 s_1/l + n\theta r_2{}^2 s_2/l + \text{etc.} = \Sigma n\theta r^2 s/l = (n\theta/l)\Sigma sr^2.$$

Now Σsr^2 is (486) the moment of inertia of a uniform thin lamina of the shape and size of the section of the cylinder and (304) of surface density unity (called for shortness the moment of inertia of the section), about an axis through its centre perpendicular to its

plane, and (490, Ex. 11) if a is the radius of the cylinder, is equal to $\pi a^4/2$. Hence $T=n\theta\pi a^4/2l$, and $n=2Tl/\theta\pi a^4$.

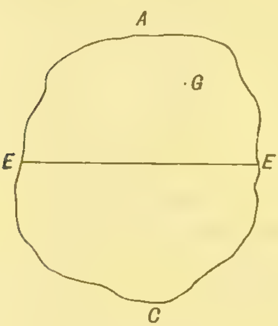
Hence also the torsion produced in a wire is directly proportional to the twisting couple and to the length of the wire, and inversely proportional to the rigidity and to the fourth power of the radius. The proportionality of the angle of torsion to the twisting couple was discovered experimentally by Coulomb, and is called Coulomb's law.

(7) Express the modulus of torsion (M_t) of a wire (552) in terms of its dimensions and its rigidity; also the rigidity in terms of the modulus.

Ans. $M_t=n\pi a^4/2$, a being the radius of the wire; $n=2M_t/\pi a^4$.

(8) A uniform straight beam, with one end fixed, is slightly bent by a force F applied at the other end normally to its length and in the plane of bending, F being so great that the weight of the beam may be neglected. Find the flexural rigidity (552) of the beam in the plane of bending.

Since the beam is uniform, and is but slightly bent, the strain produced in any small portion of it may be taken to be that of 277, Ex. 4.—Let $AECE'$ be any transverse section of the beam. Then



the part of the beam between this section and the free end is in equilibrium under the force F, the normal stress over $AECE'$, due to the longitudinal strain, and the shearing stress over $AECE'$, due to the shearing strain. Let EE' be the intersection with $AECE'$ of the neutral surface. Then at any point G, distant d from EE', there is a longitudinal strain in a direction normal to $AECE'$, the elongation being d/ρ, where ρ is the radius of curvature of longitudinal lines in the neutral surface and therefore, since the bending is slight, of all longitudinal lines. Hence, if S is the intensity of the longitudinal stress at G, and M is Young's modulus for the beam (552), $M=S/(d/\rho)$, and therefore $S=Md/\rho$. If s is an indefinitely small area surrounding G, the resultant stress on this area is Msd/ρ. The moment of this resultant stress about EE' is therefore Msd^2/ρ.

Now the whole area $AECE'$ may be divided into an indefinitely large number of indefinitely small portions. Hence the moment about EE' of the normal stress over the whole surface $AECE'$ is

$$\Sigma(Msd^2/\rho)=(M/\rho)\Sigma sd^2,$$

the summation applying to all the small areas into which $AECE'$ is divided. Now Σsd^2 is (486 and 556, Ex. 6) the moment of inertia of the surface $AECE'$ about EE'. Calling this I, we find the moment about EE' of the normal stress on $AECE'$ equal to MI/ρ.

The shearing stress on $AECE'$ being tangential has no moment about EE'.

If the distance from $AECE'$ of the free end of the beam be δ, the moment of F about EE' is $F\delta$.

The portion of the beam between $AECE'$ and the free end is thus in equilibrium under the two moments $F\delta$ and MI/ρ. Hence (500)

$$F\delta=MI/\rho, \text{ and } F\delta\rho=MI.$$

To determine I we must know the position in the beam of the neutral surface. We have seen that, s being any small portion of a transverse section, the resultant stress on it normal to the transverse section has the magnitude Msd/ρ. Hence the resultant normal stress over the whole section is

$$\Sigma Msd/\rho=(M/\rho)\Sigma sd.$$

Now the bending being slight, the direction of this resultant longitudinal stress is perpendicular to the directions of the other acting forces. Hence for equilibrium this resultant stress must be zero, and therefore $\Sigma sd=0$. Hence (403) the line EE', distant d from the little area s, passes through a point which is the centre of mass of the section $AECE'$ (527, Ex. 2), and therefore the neutral surface is the surface passing through the centres of mass of the transverse sections of the beam, and normal to the plane of bending. (That line of the neutral surface which passes through the centres of mass of the sections of the beam is called the *elastic central line.*)

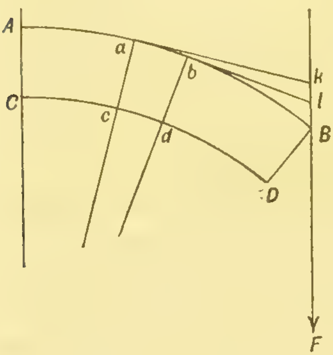
For a uniform beam, I and consequently MI have thus the same values at all sections. Now $1/\rho$ is the curvature of the beam at any section such as $AECE'$, and $F\delta$ is the couple producing this curvature. Hence $F\delta\rho$ is the modulus of flexural rigidity for such section. Hence the flexural rigidity has the same value at all sections, viz. MI, the product of Young's modulus for the material

into the moment of inertia of a transverse section about the line in which the section intersects the neutral surface.

It follows that the curvature of such a beam will be different at different sections, being equal to $F\delta/MI$, and therefore proportional to δ.

We can now calculate the flexural rigidity of a beam of given section. Thus let the transverse section be rectangular, its sides being a and b. Then (490, Ex. 4) the moment of inertia of a transverse section about an axis parallel to the sides a, in its plane, and through its centre of mass, is $ab^3/12$. Hence the flexural rigidity in a plane normal to the sides a is $Mab^3/12$, where M is Young's Modulus for the beam.

(9) A uniform straight horizontal beam of length L has one end fixed, and is slightly bent in a vertical plane by the weight F of a body attached to the other end. Find the distance through which the free end will be lowered.



Let the unstrained beam be divided into an indefinitely large number of transverse slices of thickness t, and let $abdc$ be one of these slices in the strained state. The transverse sections ac and bd will intersect one another in a horizontal line. Let θ be the inclination of ac to bd. Let ak and bl, tangents at a and b respectively, intersect a vertical line through B in k and l respectively. Then kl is the lowering of B due to the strain of $abdc$. The whole lowering of B will be the sum of the amounts of the lowering due to the strains of the various slices. Hence, if kl be denoted by λ, the total lowering of B will be $\Sigma\lambda$. Now the angle between ak and bl is θ. Hence, since the bending is slight, if the distance of the slice $abdc$ from the free end be denoted by δ, we have

$$\lambda = \delta\theta.$$

Now (277, Ex. 4) $\qquad t = \rho\theta.$

Hence $\qquad \lambda = \delta t/\rho.$

Also (556, Ex. 8) $\qquad MI/\rho = F\delta.$

Hence $\qquad \lambda = Ft\delta^2/MI.$

Hence also the total lowering of the free end

$$\Sigma\lambda = \Sigma(Ft\delta^2/MI) = (F/MI)\Sigma t\delta^2,$$

the summation extending to all the slices of thickness t into which the beam of length L is divided. Now (486) $\Sigma t\delta^2$ is the moment of inertia of a uniform thin rod of length L, and linear density unity, about a normal axis through its end point, and is therefore (490, Ex. 1) equal to $L^3/3$. Hence the whole lowering $l = \Sigma\lambda = FL^3/3MI$.

The flexural rigidity M_f, of the beam, may therefore be expressed in terms of the lowering l. For we have

$$M_f = F\delta\rho = MI = FL^3/3l.$$

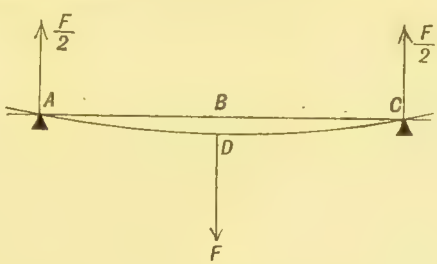
Young's modulus may be similarly expressed by the equation :

$$M = FL^3/3lI.$$

If the beam have a rectangular transverse section of breadth a and depth b, as in Ex. 8, $I = ab^3/12$. Hence, in this case, the lowering $l = 4FL^3/Mab^3$ and $M = 4FL^3/lab^3$.

(10) A uniform straight beam of length L is supported (not fixed) at the ends horizontally, and weighted at its middle point with a body of weight F. Find the amount of the lowering of the middle point, the bending being slight.

Obviously this case is the same as if the middle point of the beam were fixed and its ends acted upon by upward forces equal to $F/2$.



For the beam is in equilibrium under the force F and the equal reactions of the supports, which, since the bending is slight, may be considered vertical. Hence the lowering will be obtained from the result of Ex. 9 by putting $L/2$ for L and $F/2$ for F, which gives

$$l = FL^3/48MI, \quad M_f = FL^3/48l, \quad \text{and} \quad M = FL^3/48lI.$$

If the beam be of rectangular section, breadth $= a$, and depth $= b$,

$$l = FL^3/4Mab^3 \quad \text{and} \quad M = FL^3/4lab^3.$$

557. *Kinetics of Elastic Solids.*—The motion of the parts of an elastic solid relative to one another is to be determined by dividing it into small portions and applying the general equations of motion (550) to these portions, the forces acting on them at any instant being the stresses which must act across their bounding surfaces to produce the state of strain which they may have at that instant, and the stresses due to viscosity, together with the external forces.

In general the investigation of the motion of elastic solids which have been strained and then "let go," requires more mathematical power than this book presupposes. We may however solve one simple problem.

558. *Example.*

A uniform cylindrical body (moment of inertia about the axis $=I$) is hung by means of a wire (length $=l$, radius $=a$, rigidity $=n$) whose axis is in the same straight line with the axis of the cylinder. The cylinder, after having been turned about its axis through an angle involving a torsional strain in the wire which is within the limits of perfect elasticity, is let go. Find the time of oscillation of the cylinder (neglecting viscosity), and show how the rigidity of the wire may be determined by observation of the time of oscillation.

The cylinder may be considered to be a rigid body acted upon in a horizontal plane by no forces except the shearing stresses on its upper end where it is attached to the wire. We found (556, Ex. 6) that the couple necessary to twist a wire of length l, radius a, and rigidity n, through an angle θ is $n\pi\theta a^4/2l$. Hence at the instant at which the cylinder is turned θ radians from the position in which the wire is without torsion, the moment of the stresses exerted by the lower end of the wire on the cylinder about its axis will have this magnitude. Its direction will be such as to turn the cylinder towards the position in which the wire is without torsion. If therefore α is the angular acceleration produced in the cylinder by these stresses, we have (493)

$$\alpha=\pi n\theta a^4/2lI.$$

Hence $a \propto \theta$; and therefore for any point of the cylinder distant r from the axis, $ar \propto \theta r$, *i.e.*, the component acceleration of the point in the direction of its path varies as its displacement (measured along its path) from its mean position, that occupied when the wire is without torsion. The motion of each point of the cylinder is therefore (164) simple harmonic. Hence the cylinder will oscillate about the position in which the wire has no torsion, the rate of change of speed of each of its points when at unit distance (measured along its path) from its mean position being

$$ar/\theta r = a/\theta = \pi n a^4/2lI.$$

Hence, if t be the time of a complete (double) oscillation,

$$t = 2\pi(2lI/\pi n a^4)^{\frac{1}{2}}.$$

Hence also $n = 8\pi lI/t^2a^4$. If therefore t be observed, n may be determined.

The time of oscillation may be expressed in terms of the moment T of the couple exerted on the body for the angle of torsion, θ. For, as (556, Ex. 6) $T/\theta = n\pi a^4/2l$ we have

$$t = 2\pi(I\theta/T)^{\frac{1}{2}}.$$

It may also be expressed in terms of the modulus of torsion (M_t). For as $M = Tl/\theta$, we have

$$t = 2\pi(Il/M_t)^{\frac{1}{2}}.$$

559. *Work done during Strain.*—The work done during a strain can be best studied by considering a cube of the body subjected to the strain, whose edges have the directions of the rectangular axes by reference to which the strain is specified. Let $ODEF$ (Fig. of 533) be such a cube subjected to a stress (P, Q, R, S, T, U), and let it undergo a small elongation e alone. (We use the symbols of 283 and 535 to specify strain and stress.) The only stresses in the direction of e (that of the x axis) are P, T, and U, and the equal and opposite stresses on the opposite sides of the cube. The distance of the places of application of the two opposite stresses P is changed by the elongation, by the amount el, if l is the edge of the cube. Hence work is done equal to $Pl^2 . el$ or Pel^3. The places of application of the pair of stresses T, and of the pair U,

are not moved relatively to one another by the elongation. Hence no work is done by either. Hence the whole work done during the elongation e is Pel^3.

Similarly, if the body undergo small elongations f or g alone, the whole work done will be Qfl^3 or Rgl^3 respectively.

If now the body undergo the small shear a alone, seeing that we may regard it as having the direction of the y or of the z axis, *i.e.*, as being a shifting of planes parallel to the xy plane in the direction of the y axis, or of planes parallel to the xz plane in the direction of the z axis, the pairs of stresses R, T', S', S, Q, and U' may be in the direction of motion. Now the pair of stresses R, and the pair Q, are longitudinal stresses, and the distances of their places of application are not changed by the shear. Hence they do no work. Also the places of application of the pair of stresses T', and of the pair U', undergo no change of distance. Hence T' and U' do no work. If the shear be a shifting of planes parallel to the xy plane in the direction of the y axis, the places of application of the pair of stresses S experience a relative displacement in the direction of S, of the amount al, while the places of application of the pair S' undergo no change of distance. And if the shear be a shifting of planes parallel to the xz plane in the direction of the z axis, the places of application of the pair S' experience a relative displacement in the direction of S' of the amount al, while those of the pair S undergo no change of distance. Hence, S being equal to S', the work done in either case and therefore the whole work done during the shear a, is Sal^3.

Similarly during small shears b or c, occurring alone, the work done would be Tbl^3 or Ucl^3 respectively.

Now the translation or rotation which may accompany any strain do not change the distances of the places of application of any of the pairs of stresses P, Q, R, etc., and therefore they do not involve the performance of any

work by these stresses. Also the work done during a small strain (e, f, g, a, b, c) is the sum of the amounts of work done during each component alone. Hence the whole work done throughout a cube of edge l, subjected to a homogeneous stress (P, Q, R, S, T, U) during a small strain (e, f, g, a, b, c) is

$$(Pe+Qf+Rg+Sa+Tb+Uc)l^3.$$

Hence also the work done throughout the body per unit of volume is

$$Pe+Qf+Rg+Sa+Tb+Uc;$$

and the whole work done, if the body have the volume v, is

$$(Pe+Qf+Rg+Sa+Tb+Uc)v.$$

560. This amount of work is equal to that done on the body by the stresses on its bounding surface. For if the body be divided into indefinitely small cubes, the work done by the stress on any side of any cube is equal to that done on the contiguous side of the neighbouring cube and of opposite sign. Hence the sum of the amounts of work done on all internal surfaces is zero; and there remains only the work done by the stresses on those faces of cubes which are parts of the bounding surface of the body.

561. Let a body subjected to a stress (P, Q, R, S, T, U) undergo a small strain (e, f, g, a, b, c), and let its stress after the strain be (P', Q', R', S', T', U'). Then since, by Hooke's law, the stress is proportional to the strain, the mean stress is one half the sum of its initial and final values. Hence the work done is equal to

$$\{(P+P')e+(Q+Q')f+(R+R')g+(S+S')a$$
$$+(T+T')b+(U+U')c\}v/2.$$

If initially the body is in a state of no strain, and therefore of no stress, the work done is thus

$$(P'e+Q'f+R'g+S'a+T'b+U'c)v/2.$$

562. If the body be perfectly elastic, and if the strain be conducted so slowly that no change of temperature results, and no effect of viscosity is appreciable, then the stresses called into play depend only on the configuration of the body, and it thus constitutes a conservative system. Hence the potential energy of the body in its final configuration is equal to the work done in producing it.

563. If the body be perfectly elastic, and if the strain be not effected with infinite slowness, the stresses at the various stages of the strain are not dependent wholly upon the configuration, but depend also upon the varying temperature and upon the viscosity. Hence in this case the body does not behave as a conservative system, and the final potential energy is less than the work done in producing the change of configuration, the difference being the amount expended in the production of heat.

564. If the body be not perfectly elastic, then, even if the change of configuration be effected with infinite slowness, the stress required to produce a strain is not equal to that required to maintain it. Hence in this case also the body does not behave as a conservative system, and the final potential energy is less than the work done.

565. The potential energy of a body strained to the extreme limit of perfect elasticity is called the *resilience* of the body for that kind of strain. It is usually measured in gravitational units, and expressed per unit mass of the body. It is obvious that the resilience of a body thus expressed is equal to the height to which the body would be lifted if an amount of work equal to the resilience were done in lifting it. The term resilience is also used by some writers as synonymous with elasticity.

566. *Statics of Fluids* (*Hydrostatics*).—A fluid is a body which possesses no degree of elasticity of shape, *i.e.*, its shape may be changed by a stress of any magnitude

however small, and no stress is required to maintain the strain thus produced, the body exhibiting no tendency to return to its initial shape when the distorting stress is removed. In consequence of the viscosity of fluids however, a finite stress is necessary to produce a change of shape, if the change is to be effected with finite rapidity.

567. All fluids are perfectly elastic for condensation strains. But they differ greatly in compressibility. *Liquids* are fluids whose compressibility is small; *gases*, fluids whose compressibility is great.

The compressibility of most liquids is so small that the properties of the ideal liquid, a liquid of constant density, are approximately those of many real liquids. Hooke's law applies to the condensation of liquids up to the highest pressures to which they have been subjected. In discussing liquids, however, we shall assume their density to be invariable.

The relation of the pressure to the volume of a given mass of gas kept at constant temperature is approximately expressed in Boyle's law, which states that the pressure is inversely proportional to the volume, and therefore directly proportional to the density. All gases at sufficiently high temperatures follow Boyle's law with considerable accuracy through extensive ranges of pressure. But the lower their temperature the greater their deviation from it. We may take as the ideal gas one which follows this law, and in dealing with gases we assume it to hold.

568. The distinctive property of fluids, that the maintenance of a shearing strain requires no stress, may obviously be expressed thus:—Provided the parts of a fluid body are not moving relatively to one another, the shearing stresses at all points of the fluid are zero, or the stresses at all points on all surfaces through the points are normal.

569. *Stresses in Fluids.*—The stresses of fluid bodies are usually pressures, though in certain cases they may be tensions. The centre of stress in the case of a fluid is thus usually spoken of as a centre of pressure.

The stress throughout a fluid, which is in equilibrium and is not acted upon by external forces throughout its mass, is homogeneous (522). For (1) any hemispherical portion of it is in equilibrium; and the pressures on the small portions into which its curved surface may be divided being all normal to these portions, and therefore passing through the centre of the sphere, their resultant also passes through that point. Hence also the resultant of the pressure on the plane surface passes through its centre; and the pressure over it is therefore uniform. Also (2) any cylindrical portion, with ends normal to the axis of the cylinder, is in equilibrium, and the pressures on the curved portion of its surface being normal to the axis, the pressures on its ends must be equal and opposite. Hence the pressures on parallel surfaces are equal.

570. *Specification of Fluid Pressure.*—The stress throughout a fluid in equilibrium and not acted on by external forces being homogeneous, the results of 531-535 apply to the case of a fluid in this state. In the case of a fluid however the equations of 534 are much simplified by the absence of shearing stresses (568), and thus become

$$F_x = Pl; \; F_y = Qm; \; F_z = Rn.$$

Since F is a fluid pressure, it is normal to ABC. Hence its direction cosines are l, m, n, and $F_x = Fl = Pl$. Hence also $F = P$. Similarly $F = Q$ and $F = R$.

If therefore a fluid be in equilibrium and be not acted upon by external forces, the pressures at all points across all surfaces through these points are the same.

If it be acted upon by external forces (537) the pressures at any one point across all surfaces through that point are

the same, or, as it is usually put, the pressure at any point is the same in all directions.

The pressure at any point of a fluid in equilibrium is therefore specified by one numerical datum.

571. *Equal Transmission of Pressure.*—If P and P' be the pressures on the ends (normal to the axis) of a cylinder of unit section, of any length and in any direction, and if F be the sum of the components in the direction of the axis of the external forces acting on the cylinder, then for equilibrium

$$P' - P = F.$$

Hence, if P be increased by any amount, P' becomes increased by the same amount. This result is often called the Principle of the equal transmission of pressure.

572. *Surfaces of Equal Pressure* in a fluid acted upon by external forces and in equilibrium are surfaces at all points of which the pressure is the same.

Lines of force in a fluid acted upon by external forces are lines whose directions at all points coincide with the directions of the resultant external force at those points.

573. Surfaces of equal pressure are at all points normal to lines of force. For the resultant external force on a small cylinder of the fluid with ends normal to its axis, and so placed that the pressures on its ends are equal, can have no component in the direction of the axis.

574. If the external forces are central forces (338), and the various points of the fluid have therefore potentials (355-6), the resultant force at a point must be normal to the equipotential surface through the point (359). Hence surfaces of equal pressure coincide with equipotential surfaces.

575. In that case also (356) the resultant external force on unit mass of the fluid at any point is equal to the rate of change of potential per unit of distance in its direction. Now, if the fluid between two surfaces of equal pressure, indefinitely near one another, be divided by lines of force into columns of equal section, the differences of pressure on the ends being the same for all, and all being in equilibrium, the resultant external forces acting on all must be the same. Let F be the resultant external force on any column, m its mass, l its length and V and V' the potentials at its ends. Then $F/m=(V'-V)/l$. Hence, the difference of potential between the ends being the same for all, the ratio of the mass to the length and therefore of the mass to the volume must be the same for all. And therefore surfaces of equal pressure are also surfaces of equal density.

576. In the case of heavy fluids, the attraction of the earth is the external force. Hence in that case level or horizontal surfaces are surfaces of equal pressure.

The free surface of a heavy liquid in equilibrium, being exposed to the pressure of the atmosphere, is therefore a horizontal surface throughout the region in which the pressure of the atmosphere has the same value.

577. *Variation of the Pressure of Fluids acted upon by External Forces.*—Let F be the resultant external force acting on each unit of volume of the fluid, in one of the columns of 575, s being the area of either of its ends, l its length, and P and P' the intensities of the pressures on its ends. Then

$$(P'-P)s=Fls,$$

and

$$(P'-P)/l=F.$$

Hence the resultant force on unit volume of the fluid is equal to the rate of change of pressure in its direction per unit of distance.

578. If the external forces are derivable from a potential, we have also (356), V' and V being the potentials at the ends of the column at which the pressures are P' and P respectively, and ρ being the density,

$$(V' \sim V)/l = F/\rho.$$

Hence $$(P' - P) = (V' \sim V)\rho.$$

If gravitational attraction is the only external force, we have therefore, with the convention of 361, since the external force is directed from the end of smaller to the end of greater pressure,

$$P' - P = \rho(V' - V).$$

Now in this case $$V' - V = gl.$$

Hence $$P' - P = \rho g l;$$

and therefore the rate at which pressure increases per unit of distance in a direction normal to surfaces of equal pressure in a heavy fluid is equal to ρg.

579. In the case of liquids ρ is a constant. Let P_0, P_1, P_2, etc., P be the pressures at a series of surfaces of equal pressure indefinitely near, let l_1, l_2, etc., be the intercepts between these surfaces of a line of force, and let the surfaces whose pressures are P_0 and P be so near that g may be considered constant, then

$$P_1 - P_0 = \rho g l_1, \quad P_2 - P_1 = \rho g l_2, \quad \text{etc.}$$

Hence, if L be the length of the line of force extending from any point of the surface whose pressure is P to that of which the pressure is P_0, we have by addition

$$P - P_0 = \rho g(l_1 + l_2 + \text{etc.}) = \rho g L.$$

Gravitational attraction being the only external force acting throughout the mass of the fluid, the surfaces of equal pressure are horizontal surfaces and the lines of force are vertical lines. Hence the difference of pressure between two points of a heavy liquid is equal to their difference of level multiplied by ρg, and therefore to the

weight of a column of the liquid whose length is the difference of level and whose section is unity.

If Π is the pressure of the atmosphere at the free surface of a heavy liquid, the pressure at any point at depth L is thus $\Pi+\rho gL$, which may be written $\rho g(L+L')$, provided $L'=\Pi/\rho g$, *i.e.*, provided L' is the length of a column of the liquid of unit section whose weight is equal to Π.

The determination of the resultants and centres of the pressures on the surfaces of bodies immersed in heavy liquids is of great practical importance. The reader will find, on looking back to 527, Exs. 2-5, and 529, Exs. 2-7, that examples of such determinations have already been given in considering resultants and centres of stress.

580. In the case of gases kept at a constant temperature we have (567) $\rho=kP$, where k is a constant and ρ the density of a gas at the point at which its pressure is P. Hence

$$P'-P=kP(V'-V),$$

and

$$P'=P[1+k(V'-V)].$$

Let P_0, P_1, P_2, etc., P be the pressures at a series of $n+1$ surfaces of equal pressure indefinitely near, and so chosen that the differences of potential of neighbouring surfaces are the same, and let V, V_0 be the potentials of the surfaces whose pressures are P, P_0. Then

$$P_1=P_0[1+k(V-V_0)/n],$$

$$P_2=P_1[1+k(V-V_0)/n],$$

$$=P_0[1+k(V-V_0)/n]^2,$$

etc.,

$$P=P_0[1+k(V-V_0)/n]^n.$$

Hence, if n be made indefinitely great, we have (392)

$$P=P_0e^{k(V-V_0)},$$

where e is the base of Napier's Logarithms.

Gravitational attraction being the only external force acting throughout the mass of the gas, and the volume of the gas under consideration being so small that g may be considered constant, we have

$$V - V_0 = -gh,$$

where h is the height of the point whose potential is V above that whose potential is V_0. Hence

$$P = P_0 e^{-kgh},$$

and

$$h = \frac{1}{kg}(\log_e P_0 - \log_e P).$$

The difference of level (h) of two places might be found by means of this expression, from observations of the pressure of the atmosphere, were it not for the variation of temperature, k being constant only at constant temperature.

It is obvious that since $P = \rho/k$, $1/k$ is equal to g times the length of a column of the given gas of uniform density ρ and of section unity, whose weight is equal to P. It is therefore equal to g times the height which an atmosphere of the gas would have if its density were the same throughout its whole extent as at the earth's surface. This height is consequently often called the "height of a homogeneous atmosphere" or the "pressure-height" of the given gas for the temperature to which the given value of k applies. If this height be denoted by H, since $1/k = gH$, we have

$$P = P_0 e^{-\frac{h}{H}},$$

and

$$h = H(\log_e P_0 - \log_e P).$$

The value of H for any gas depends only on its nature and temperature and on the value of g. For dry atmospheric air at 0° C. in the latitude of Paris it is $7{\cdot}990 \times 10^5$ cm.

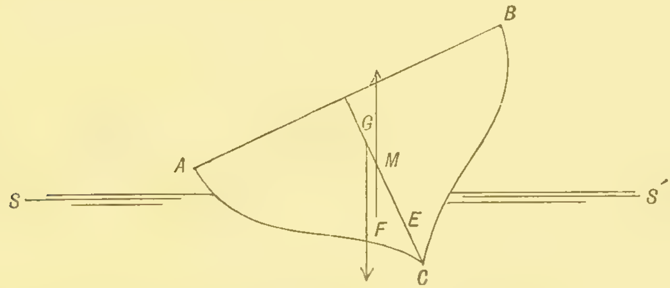
581. *Archimedes' Principle.*—If a body be wholly or partially immersed in a heavy fluid, the resultant of the pressure over its surface is a single force acting vertically upwards through its centre of mass and equal to the weight of the fluid displaced. For a portion of the fluid having the same position, shape, and size as the given body or the part of it which displaces fluid, would be in equilibrium under its own weight and the resultant pressure on its surface, which, since the pressure at a point of a heavy fluid varies only with its depth beneath or height above any chosen level surface, must be the same as the resultant pressure on the body.

Various methods for determining specific gravity (304) are based on Archimedes' Principle. For descriptions of them readers are referred to books on Experimental Physics.

582. *Equilibrium of a Floating Body.*—It follows, from 581, that a body floating at the surface of a heavy liquid will be in equilibrium provided (1) the centres of mass of the body and of the displaced liquid are in a vertical line, and (2) the weight of the body is equal to that of the displaced liquid.

583. *Stability of the Equilibrium of a Floating Body.*—The general discussion of the stability of the equilibrium of a floating body is beyond the scope of this book. But in the important special case of a homogeneous rigid cylinder, of any section, for angular displacements about its axis, the condition of stability admits of simple expression.—Let ABC be a transverse section of such a cylinder, through its centre of mass G; and let E be the centre of mass of the portion beneath the surface SS' of the liquid, and therefore of the displaced liquid, in the position of equilibrium, in which the line GE is obviously vertical. Also let F be the centre of mass of the submerged portion when the cylinder has been rotated through a small angle about a longitudinal axis, M being

the point in which a vertical line through F intersects GE. Then the cylinder is acted upon by equal and opposite vertical forces through G and F; and it is obvious that if the point M be above G these forces will tend to diminish the angular displacement and to bring



the cylinder back to the position of equilibrium; whereas, if M be below G, they will tend to increase the displacement. In the former case therefore the equilibrium is stable, in the latter unstable. The point M is called the metacentre. The equilibrium is therefore stable, provided the metacentre be above the centre of mass. This result applies to the rolling of a ship so built and laden that G, E, and F are in the same plane.

584. *Kinetics of Fluids (Hydrokinetics.)*—When the parts of a fluid move relatively to one another, frictional shearing stresses make themselves manifest. If, *e.g.* a cylindrical vessel, with its axis vertical, and containing a liquid, be made to rotate uniformly about its axis, the liquid will be found after a time to be rotating with the vessel, and if the vessel be now brought to rest the motion of the liquid gradually subsides. Hence any cylinder of the liquid coaxial with the vessel is acted upon by stresses having tangential components when the liquid outside it is in motion. For otherwise that cylinder must remain at rest or in uniform motion.

In many important practical cases however the effect of these shearing stresses is small and may be neglected;

and as the consideration of the motion of fluids exhibiting tangential stresses is attended with great difficulty, we restrict our attention to these cases.

When the parts of a fluid move relatively to a solid with which they are in contact, the frictional stresses called into play may be very considerable, and require to be taken into account, if results of practical value are to be obtained.

585. If the stresses at a point of a moving fluid on all planes through the point are normal, they have also the same intensity.

For if we consider a tetrahedron, such as that of 532, we have, as the equation of its motion in the direction of Ox, using the symbols of 532,

$$P \cdot OBC - Fl \cdot ABC + X = m\bar{a},$$

where X is the component of the resultant external force, m the mass, and $\bar{a}$ the component acceleration of the centre of mass in the x axis. Now X is proportional to the mass, and therefore to the volume, of the tetrahedron. If therefore (537) the tetrahedron be indefinitely small, both X and m may be neglected relatively to P and F. Also we have $OBC = ABC \cdot l$. Hence $P = F$.

586. *Equations of Motion.*—The motion of a fluid under given forces may be determined by applying the general equations of the motion of extended bodies and expressing in equations the conditions imposed by the distinctive peculiarities of fluids. Of these equations there are two. The first expresses the relation which holds between the pressure and the density of the fluid. In the case of a gas at constant temperature it is $\rho = kP$: and in that of a liquid, $\rho =$ const. The second is the *equation of continuity* which expresses in mathematical language the general law that a fluid in motion is always a continuous mass. The employment of these equations

however in the solution of problems is beyond the scope of this book.

587. *Steady Motion.*—In general the velocity of the fluid particles passing through a given point in space varies with time. When at each point in space through which fluid is passing the velocity of the fluid is constant both in magnitude and direction, the motion is said to be steady.

The paths of the particles of a fluid which is moving steadily, are lines of motion, *i.e.*, lines whose directions at all points are the directions of the motion of the fluid at those points. They are therefore called *stream lines.*

588. *Equation of Energy.*—We may obtain, as being simple and important, the equation of energy applicable to cases of the steady motion of heavy fluids.

Consider a tube whose curved surface is bounded by stream lines, and whose ends A and B are small and normal to the stream lines and at heights h and h' above any chosen horizontal plane of reference. Let p be the pressure, v the speed, and ρ the density of the fluid at A, p', v', and ρ' the corresponding values for B. Let s, s' be the areas of the normal sections of the tube at A and B respectively. The masses of the fluid entering the tube at A and leaving it at B in a time τ are $\rho s v \tau$ and $\rho' s' v' \tau$ respectively. Hence the kinetic energy entering at A and the amount leaving at B in time τ are $\rho s v \tau \,.\, v^2/2$ and $\rho' s' v' \tau \,.\, v'^2/2$ respectively; and the gravitational potential energy entering at A and that leaving at B are $\rho s v \tau g h$ and $\rho' s' v' \tau g h'$ respectively. The work done by the resultant pressure at A on the fluid entering during τ is $ps \,.\, v\tau$; the amount done by the fluid against external pressure on leaving at B is $p's' \,.\, v'\tau$; hence the energy of the fluid in the tube is increased and diminished by these respective amounts. Work will also in general be done by the fluid in the tube against fric-

tion or other forms of non-conservative force, say to the amount W.

As the motion is steady, and the energy of the tube therefore constant, the energy entering it must be equal to the energy leaving it, both convection and transformation being taken into account. Hence

$$\rho s v \tau\left(\frac{v^2}{2}+gh+\frac{p}{\rho}\right)=\rho' s' v' \tau\left(\frac{v'^2}{2}+gh'+\frac{p'}{\rho'}\right)+W.$$

If follows also from the steadiness and continuity of the motion, that the mass of fluid in the tube is constant. Hence the mass entering in time τ must be equal to the mass leaving, and we have

$$\rho v s \tau=\rho' v' s' \tau,$$

the equation of continuity in this case. Hence, dividing one side of the equation by the one, and the other by the other, of these equal quantities, we have

$$\frac{v^2}{2g}+h+\frac{p}{\rho g}=\frac{v'^2}{2g}+h'+\frac{p'}{\rho' g}+\frac{W}{\rho s v \tau g}.$$

All the terms in this equation have the dimensions of a length or height. The height h of a point in a moving fluid above the plane of reference is called by hydraulic engineers the *head* of the point. The heights to which $p/\rho g$ and $p'/\rho' g$ are equivalent, viz., the heights of the columns of the fluid of unit section and uniform density whose weights would be equal to p and p' respectively are called pressure heads. The heights to which $v^2/2g$ and $v'^2/2g$ are equivalent, viz., the heights above A and B from which the fluid particles would need to fall freely, in order to gain the velocities v and v' respectively, are called velocity heads, and the height to which $W/\rho s v \tau g$ is equivalent is called the head wasted, or waste of head. Putting the equation in the following form, in which w stands for waste of head,

$$h-h'=\frac{v'^2}{2g}-\frac{v^2}{2g}+\frac{p'}{\rho' g}-\frac{p}{\rho g}+w,$$

we see that it may be expressed as follows: In passing from any one point to any other in a stream line the loss of head is equal to the gain of velocity and pressure heads together with the waste of head.

589. *Application to Liquids.*—In the case of a liquid the density is constant. Hence $\rho=\rho'$, and

$$h-h'=(v'^2-v^2)/2g+(p'-p)/\rho g+w.$$

590. *Examples.*

(1) Find how the pressure is related to the velocity at two points of a stream tube of variable section, between which the difference of level and the waste of head are negligible.

As $h-h'=0$ and $w=0$, we have

$$v'^2/2+p'/\rho=v^2/2+p/\rho=\text{constant}.$$

Hence where the section of the tube is small and therefore the velocity is great the pressure must be small, and where the velocity is small the pressure must be great. If, therefore, liquid flows steadily through a horizontal frictionless pipe of variable section, the pressure must be smaller at sections of small area than at sections of greater area though the rush of liquid through the smaller sections would lead one at first sight to expect the opposite.

If a jet of water is flowing from an orifice in a plane surface and a disc (not too heavy) is placed near the surface approximately parallel to it and opposite to the jet, the water is made to flow radially through the narrow space between the disc and the surface; its stream tubes increase in section from the orifice outwards, and the pressure of the water therefore increases from the orifice outwards. But at the edge of the disc it is atmospheric. Hence the average pressure exerted by the water on the disc is less than that of the atmosphere. On the other side of the disc the pressure is atmospheric. Hence the disc is driven towards the orifice, though at first sight one would expect it to be driven from it.

(2) A vessel is kept filled to a constant level (h above the horizontal plane of reference) with liquid which escapes through a

small orifice in a thin part of one of its sides. Find the speed of efflux after the motion has become steady.

Consider a stream tube extending from the surface of the liquid through the orifice. At the surface, if the orifice is sufficiently small the velocity of the liquid will be negligible. Hence $v=0$. The pressure will be that of the atmosphere Π. Hence if v' is the velocity at any other point of the stream tube, where p' and h' are the pressure and the head,

$$h-h'=v'^2/2g+(p'-\Pi)/\rho g+w.$$

If we assume that at the orifice the pressure is that of the atmosphere and that the waste of head is zero, we have for the velocity of the liquid at the orifice

$$v'^2=2g(h-h'),$$

h' being the height of the orifice above the plane of reference. This result is called Torricelli's Theorem. Even if the above assumptions were warranted we could not use this value of the velocity in order to calculate the rate of efflux, because the jet diminishes in diameter from the orifice outwards, and the stream lines at the orifice are therefore not normal to it.

The assumptions are not warranted however. There will be some waste of head due to friction at the orifice; and as the jet diminishes in diameter from the orifice outwards, the pressure (Ex. 1) must also diminish. At a short distance from the orifice the contraction of the jet ceases, that part of the jet being called the *vena contracta*. Here the pressure has ceased to diminish and may be taken to be equal to that of the atmosphere. If, therefore, h' represent the height of the *vena contracta* above the plane of reference, we have

$$h-h'=v'^2/2g+w,$$

and

$$v'^2=2g(h-h'-w).$$

As at the *vena contracta* the stream lines are normal to the section, the velocity given by this expression may be used in calculating the rate of efflux, which if S is the area of the section of the *vena contracta* will be equal to $\rho S v'$. In order to find v' from the above expression, however, w must be determined by experiment.

(3) Find the waste of head due to friction between liquid and solid, in a pipe of uniform section.

Experiment shows that the friction (F) between a liquid and a solid in contact and in relative motion is directly proportional to the density of the liquid, the area S of the surface of contact and the square of the relative velocity v. Hence

$$F = f\rho S v^2,$$

where f is a constant for a given liquid and a given solid with its surface in a given state. If l be the length of the pipe and r its radius, $S = 2\pi r l$. Hence

$$F = 2\pi f \rho r l v^2.$$

The work done against friction during a time τ will be

$$W = 2\pi f \rho r l v^2 \times v\tau.$$

Dividing by $\rho s v\tau$, which is equal to $\pi\rho r^2 v\tau$, we obtain the waste of head

$$w = 2flv^2/r.$$

(4) A liquid flows from a reservoir through a pipe of length l, radius r, and friction constant f. Find the velocity of efflux, and the pressure at any point in the pipe.

Let the surface of the liquid in the reservoir and the lower end of the pipe be at heights h and h' above the horizontal plane of reference. At the surface of the liquid we have $v = 0$ and $p = \Pi$ as in Ex. 2. At the lower end of the pipe, as after the motion has become steady, the liquid issues in a cylindrical jet, we have $p' = \Pi$. Hence if w' represent the waste of head at the junction of the pipe and the reservoir, we have

$$h - h' = v'^2/2g + 2flv'^2/r + w'.$$

The velocity of efflux of the liquid from the pipe may be determined from this formula, if w' is either known from experiment or (as is usually the case) is negligible relative to the waste in the pipe.

As the pipe is of uniform section, v' is the velocity of the liquid at all points of the pipe. When it has been determined the pressure p'' at any point at height h'' and distance l'' from the reservoir measured along the pipe, may be determined by the formula

$$h - h'' = v'^2/2g + (p'' - \Pi)/\rho g + 2fl''v'^2/r + w'.$$

591. *Application to Gases.*—In the case of a gas the density may vary from point to point of a stream tube. Hence the equation obtained above for liquids, viz.,

$$h-h'=(v'^2-v^2)/2g+(p'-p)/\rho g+w,$$

applies only to infinitesimal portions of a stream tube of a gas. In an infinitesimal portion ρ may be considered constant, and by Boyle's Law may be put equal to either pk or $p'k$ (p and p' being ultimately equal), k being a constant for a given gas at a given temperature. For an infinitesimal stream tube therefore we have

$$\frac{p'-p}{p}=-kg\left(h'-h+\frac{v'^2-v^2}{2g}+w\right).$$

Divide any given stream tube AB of finite length into n infinitesimal portions such that the right hand side of this equation has for each, $1/n^{\text{th}}$ of the value that it has for the whole. Then if $p, p_1, p_2 \ldots p', h, h_1, \ldots h', v, v_1, \ldots v'$ are the values of the pressure, head, and velocity at the bounding sections of these portions of the tube, we have

$$p_1=p\left\{1-kg\left(h'-h+\frac{v'^2-v^2}{2g}+w\right)/n\right\},$$

$$p_2=p_1\left\{1-kg\left(h'-h+\frac{v'^2-v^2}{2g}+w\right)/n\right\}$$

$$=p\left\{1-kg\left(h'-h+\frac{v'^2-v^2}{2g}+w\right)/n\right\}^2,$$

etc.,

$$p'=p\left\{1-kg\left(h'-h+\frac{v'^2-v^2}{2g}+w\right)/n\right\}^n.$$

$$=pe^{-kg\left(h'-h+\frac{v'^2-v^2}{2g}+w\right)}.$$

Hence

$$kg\left(h'-h+\frac{v'^2-v^2}{2g}+w\right)=\log_e p-\log_e p',$$

an equation which gives the relation between the pressures, velocities, and heads at any two points of a stream tube of finite length, the constant k for the gas, and the waste of head in the tube.

592. *Examples.*

(1) Find the velocity of efflux of a gas from a vessel in which it is at pressure p, through a small orifice, assuming that the pressure just outside the orifice is atmospheric, and that the waste of head at the orifice is negligible.

In such a case the difference of head between the orifice and any point at the farther end of the vessel is negligible, *i.e.*, $h'-h=0$. Also we have $v=0$ and pressure $=p$ at that point, while at the orifice the pressure is Π, that of the atmosphere. Hence

$$v'^2=2g\left(\frac{1}{kg}(\log_e p-\log_e \Pi)-w\right),$$

and as the waste of head is negligible,

$$v'^2=\frac{2}{k}\log_e\frac{p}{\Pi}.$$

(2) Compare the velocities of efflux of different gases under the conditions of Ex. 1, the internal and atmospheric pressures being the same for the different gases. If v_1 and v_2 are the velocities of efflux for two gases, and k_1 and k_2 their Boyle's Law constants, we have

$$v_1^2/v_2^2=k_2/k_1,$$

and therefore

$$v_1/v_2=(\rho_2/\rho_1)^{\frac{1}{2}}$$

if ρ_1 and ρ_2 are their densities at the same pressure. Hence the velocities of efflux are inversely proportional to the square roots of their densities at the same pressure.

(3) The pressure at points of a stream tube where the section is large is greater than at points where the section is small, if the waste of head between them and the difference of pressure head are negligible.

For we have, under these conditions,

$$v'^2 - v^2 = (2/k)(\log_e p - \log_e p'),$$

and therefore

$$v'^2 + (2/k)\log_e p' = v^2 + (2/k)\log_e p.$$

Hence at places where the section is small, and therefore (as the motion is steady) the velocity is great, the pressure must be small, and where the section is great the velocity must be comparatively large.

The experiment with the disc and jet of liquid described in 590, Ex. 1, may therefore be performed with a jet of gas as well.

(4) What is the effect of rotation on the motion of a projectile in air?

As a ball moves through the air, the air relatively to the ball moves around and past it. If the ball is spinning about an axis perpendicular to its direction of motion, the surface friction hinders the relative motion of the air on one side and helps it on the other. On the side on which the velocity of the air is diminished by the spin the pressure is increased, and on the side on which the velocity is increased the pressure is diminished. Hence the ball will be acted upon by a resultant force due to air pressure, directed from the side on which the rotational motion has the same direction as the motion of the centre of the ball towards the other side, and this force will affect its path. A cricket or tennis ball with a spin about a vertical axis will not move in a vertical plane, but will swerve to one side or other according to the direction of the spin. A golf ball with a spin of the proper sign about a horizontal axis, an under-spin, will rise higher, and therefore remain longer in the air, and therefore carry farther than it otherwise would.

593. *Work done during Strain.*—As tangential stresses exist in a fluid during the relative motion of its parts, the expressions obtained (559) for the work done in an elastic solid during a change of configuration apply also to fluids.

Since work is done during the straining of a fluid, in overcoming its viscosity, a fluid, like a solid, will behave

during a strain as a conservative system only if the strain be effected with sufficient slowness.

Since a fluid in equilibrium exhibits no shearing stresses, the work done against shearing stresses during a strain has no result in the form of production of potential energy.

In the case of liquids, on account of their incompressibility, a strain involves no change of volume. Hence the work done in producing a strain in their case has no result in the form of potential energy. It is wholly expended in overcoming molecular friction and results only in the production of heat. Hence Joule employed the agitation of water as a means of determining the mechanical equivalent of heat, the water employed having, after its agitation, the same potential energy as it had before.

MISCELLANEOUS EXAMPLES.

(1) A point moves in a plane curve so that its distance, s feet measured along the curve from a fixed point in it, is represented by the formula $s=25+6t^2$, where t is the time in seconds reckoned from the instant of starting. Find (*a*) the mean speed between the beginning of the 10th and the end of the twelfth second; (*b*) the instantaneous speed at the end of the 10th second; (*c*) the mean rate of change of speed between the instant of starting and the end of the 10th second; (*d*) the instantaneous rate of change of speed after any time.

Ans. (*a*) 126 ft. per sec.; (*b*) 120 ft. per sec.; (*c*) 12 ft.-per-sec. per sec.; (*d*) 12 ft.-per-sec. per sec.

(2) The breadth between the rails of a certain railway is 4 ft. 8 in. Show that in a curve of 500 yds. radius the outer rail ought to be raised about $2\frac{1}{4}$ inches for trains travelling 30 mls. an hour, that there may be no horizontal pressure on the rails.

(3) The velocity of a point moving in a given elliptic orbit is the same at a certain point, whether it describe the orbit in a time t when its acceleration is directed towards one focus, or in a time t' when its acceleration is directed towards the other focus. Show that, if $2a$ is the length of the major axis, the focal distances will be $2at'/(t+t')$ and $2at/(t+t')$.

(4) A large number of equal particles are fastened at unequal intervals to a fine string and then collected into a heap at the edge of a smooth horizontal table with the extreme one just hanging

over the edge. The intervals are such that the times between successive particles being carried over the edge are equal. Prove that if c_n be the interval between the n^{th} and $(n+1)^{\text{th}}$ particles, and v_n the velocity just after the $(n+1)^{\text{th}}$ particle is carried over,

$$c_n/c_1 = v_n/v_1 = n.$$

(5) Reduce 20 cm. per sec. to yards per hour.
Ans. 787·38.

(6) If a particle move on any smooth curve under the action of any force, and if, at any point, F be the component of this force normal to the curve and towards the concavity of the curve, the reaction of the curve on the particle towards the concavity is equal to $mv^2/\rho - F$, where ρ is the radius of curvature of the curve and v the speed of the particle.

(7) A uniform rod hangs horizontally supported by two equal vertical strings, of length l, attached to its ends. It is twisted horizontally through a very small angle so that its centre of mass remains in the same vertical line, and is then let go. Find the time of a complete (double) oscillation, neglecting the inertia of the strings.

Ans. $2\pi\sqrt{l/3g}$.

(8) A point is moving with a simple harmonic motion of amplitude a and period T. Show that, if d is its displacement from its mean position after a time t, the epoch being θ,

$$d = a\cos(2\pi t/T + \theta).$$

(9) A straight staircase consists of stairs each 1 ft. wide and 6 in. high. A smooth particle is projected from a point on one of the stairs near its edge and in the vertical plane perpendicular to the edge of each stair. Find the velocity of projection that the particle may strike the different stairs in succession at the same distance from the edge, the coefficient of restitution being 0·5.

Ans. $\sqrt{2g/3}$ feet per second, inclined 45° to the horizon.

(10) The unit of rate of change of speed being a rate of change of speed of 100 cm.-sec. units and the unit of time 1 min., show that the unit of length is a length of 36×10^4 cm.

(11) If a ball impinge successively against two adjacent sides of a rectangle, its velocity will be diminished in the ratio of $1 : e$, e being the coefficient of restitution.

(12) Two uniform solid cylinders, of weights w and w', descend from rest directly down the two faces of two smooth inclined planes, of inclinations α and α' respectively, over the common summit of which passes a thin inextensible string which goes under and round the central transverse sections of the cylinders, to which the ends of the strings are fastened. Find (*a*) the tension of the string, and (*b*) how much it will have slid along the planes at the end of any time t.

Ans. (*a*) $ww'(\sin\alpha + \sin\alpha')/3(w+w')$;
(*b*) $gt^2(w\sin\alpha - w'\sin\alpha')/2(w+w')$.

(13) A particle weighing $\frac{1}{10}$ lb. moves backwards and forwards in a straight line 3 inches long with simple harmonic motion, 25 times per second. Find the force acting on it (*a*) at the end of the range, and (*b*) at a point at one half the maximum distance from the centre.

Ans. (*a*) 616·8... pdls.; (*b*) 308·4... pdls.

(14) A particle of mass m is suspended from two points in the same horizontal line by two strings of equal length l (inclination $=\alpha$). One of the strings is suddenly cut. Find the initial change of tension of the other string.

Ans. $mg(2\cos^2\alpha - 1)/(2\cos\alpha)$.

(15) A heavy smooth tetrahedron rests with three of its faces against three fixed pegs and the fourth face horizontal. Prove that the reactions of the pegs are as the areas of the faces on which they are exerted.

(16) A point is moving with a uniform rate of change of speed of 2 ft.-sec. units. Show that, if its initial speed is 3 ft. per sec., the ratio of its final to its initial speed during the time required to traverse 4 feet of its path is 5/3.

(17) If particles are dropped from given heights upon a fixed horizontal plane, the heights being inversely as the squares of the coefficients of restitution, they all rise to the same height after reflection.

(18) A uniform lever ACB, whose arms AC and BC are at right angles to each other, is in equilibrium when AC is inclined at an angle β to the horizon. If AC be raised to a horizontal position, C being fixed, find the angle through which it will fall.

Ans. 2β.

(19) A particle of 0·1 grm. mass executes 512 simple harmonic oscillations per second, the amplitude of the oscillations being 0·25 cm. Find the maximum value of the force exerted upon it.

Ans. 258,736·1... dynes.

(20) A rope hanging over a rough horizontal cylinder carries two bodies. The mass of one is 20 lbs.; that of the other is m lbs. ($m > 20$). But the rope does not slip off the cylinder, on account of friction. If the coefficient of friction, when the rope is just on the point of slipping, is 0·4, what is the value of m?

Ans. 70·269 lbs.

(21) Q's displacement relative to P is n times as great as P's relative to O, and they are inclined at an angle θ. Show that if $\theta < \pi/2$, Q's displacement relative to O increases with n, and that, if $\theta > \pi/2$, it decreases as n increases until $n = -\cos\theta$, increasing with n for greater values of n.

(22) A particle is projected from a point on an inclined plane and after n rebounds returns to its point of projection. Prove that, if α is the inclination of the plane, β the angle between the direction of projection and the plane, and e the coefficient of restitution,

$$\cot\alpha\cot\beta = \frac{1-e^{n+1}}{1-e}.$$

(23) The time of descent of a heavy particle sliding freely from rest down a smooth inclined plane of given height varies as the cosecant of the inclination.

(24) A chain, whose weight per unit length is equal to that of 1 lb., is to be stretched between two points in a horizontal line 800 ft. apart, so that the tension at the lowest point may be equal to the weight of 1,600 lbs. Find (a) the length of chain required, and (b) the depth of its lowest point below the points of suspension.

Ans. (a) 808·32 ft.; (b) 50·24 ft.

(25) A point undergoes component displacements represented by straight lines drawn from a point within a triangle to the angular points. Show that its resultant displacement is the same as if it had undergone component displacements represented by lines drawn from the same point to the points of bisection of the sides.

(26) On the sides of a right-angled triangle squares are described, the square $BCDE$ on the hypothenuse being on the same side of BC as the triangle, the squares $CAFG$, $ABHK$ on CA, CB on the opposite side of each to the triangle. Prove that if forces represented by AB, BC, CA, BH, HK, KA, CD, DE, EB, AF, FG, GC, act on a particle, it will be in equilibrium.

(27) A particle slides from rest down the whole length of a smooth inclined plane. Prove that the distance between the foot of the inclined plane and the focus of the parabola which the particle describes after leaving the plane is equal to the height of the plane.

(28) Trucks containing each a ton of ballast are sustained upon a smooth plane of inclination α by an equal number of empty trucks upon a smooth plane of inclination β. Find the mass of a truck.

Ans. $\sin\alpha/(\sin\beta - \sin\alpha)$ tons.

(29) A right cylinder whose weight is to the diameter of its base as 3 : 4, stands on a perfectly rough inclined plane whose inclination is 45°. From the lowest point of its uppermost circular section a body is suspended whose weight is a little greater than one-sixth of the weight of the cylinder. Prove that it will overturn the cylinder.

(30) A ship sails from A to B, $\sqrt{3}$ miles N. 30° W., in 15 minutes; from B to C, 1 mile N. 60° E., in 7 minutes; from C to D, 4 miles N. 45° W., in 20 minutes; and from D to E, 4 miles N. 45° E., in 18 minutes. Show that her mean speed has been $9+\sqrt{3}$ miles per hour, and that her mean velocity has been $2+4\sqrt{2}$ miles per hour, N.

(31) $ABCD$ and $A'B'C'D'$ are two parallelograms. Show that if a particle be acted upon by forces represented by AA', $B'B$, CC', and $D'D$, it will be in equilibrium.

(32) A uniform straight plank (length $=2a$) rests with its middle point upon a rough horizontal cylinder (radius $=r$), their directions being perpendicular to each other. Supposing the plank to be slightly displaced so as to remain always in contact with the cylinder without sliding, determine the period of an oscillation.

Ans. $2\pi a/\sqrt{3gr}$.

(33) Two circles lie in the same plane, the lowest point of the one being in contact with the highest point of the other. Show that the time of descent from any point of the former to a point in the latter down the chord passing through the point of contact, is constant.

(34) Four pegs are fixed in a wall at the four highest vertices of a regular hexagon, the two lowest being in a horizontal line. Over the pegs a loop is thrown supporting a body of weight W, the loop having such a length that the angles formed by it at the lowest pegs are right angles. Find (a) the tension in the string, (b) the reactions of the two highest pegs, and (c) those of the two lowest pegs.

Ans. (a) W; (b) W, inclined 60° to the horizontal; (c) $W\sqrt{2}$, inclined 15° to the horizontal.

(35) Two points, P and Q, move in straight lines (inclination $=\theta$) with uniform accelerations a and a', and at a given instant have velocities v and v' respectively. Show that their relative velocity will be perpendicular to Q's line of motion after a time

$$(v\cos\theta - v')/(a' - a\cos\theta),$$

and will have its least value after a time

$$\frac{(av' + a'v)\cos\theta - av - a'v'}{a^2 + a'^2 - 2aa'\cos\theta}.$$

Show also that if $v/v' = a/a'$, the least value of their relative velocity will be zero.

(36) A particle of weight W is supported on a smooth inclined plane of inclination α, by means of two strings attached to fixed points in the plane and inclined at angles θ and θ' to a line of

greatest slope. Find (a) the tensions in the strings, and (b) the reaction of the plane.

Ans. (a) $\dfrac{W\sin\theta'}{\sin(\theta+\theta')}$ and $\dfrac{W\sin\theta}{\sin(\theta+\theta')}$; ($b$) $W\cos\alpha$,

(37) A given circle and a given straight line which does not cut the circle are in the same vertical plane. Show that if a tangent be drawn to the circle at its lowest point P, meeting the given line in A, and if from the given line AQ be cut off equal to AP, and if PQ intersect the circle in R, QR is the straight line of quickest descent from the given straight line to the given circle.

(38) Two equal heavy particles slide along the arc of an ellipse whose plane and major axis are vertical. They are connected by a string passing through a smooth ring at the focus. Prove that the particles will be in equilibrium in all positions.

(39) A point has three component coplanar velocities, v_1, v_2, v_3, the angles between v_3 and v_2, v_3 and v_1, v_2 and v_1 being α, β, γ respectively. Show that its resultant velocity is

$$(v_1^2+v_2^2+v_3^2+2v_2v_3\cos\alpha+2v_1v_3\cos\beta+2v_1v_2\cos\gamma)^{\frac{1}{2}}.$$

(40) If the height of a rough inclined plane be to the length as a is to $\sqrt{a^2+b^2}$, and a body of $k\sqrt{a^2+b^2}$ lbs. mass can just be supported by friction alone, required the least force acting along the plane which will draw the body up the plane.

Ans. $2kag$.

(41) Two bodies of equal weight w are tied to the ends of a fine string which passes over two pullies without mass in a horizontal line (distance $=a$). Supposing a body of weight $W(W>2w)$ to be fixed to the middle point of the horizontal portion of the string, determine how far it will descend.

Ans. $2wWa/(4w^2-W^2)$.

(42) A pendulum which would oscillate seconds at the equator would gain 5 minutes a day at the pole. Show that the ratio of the value of g at the equator to its value at the pole is 144 : 145.

(43) If there are n particles in a straight line, of masses m, $2m$, $3m$, etc., and at distances a, $a/2$, $a/3$, etc., respectively from a point in the line, the distance of the centre of mass from it is $2a/(n+1)$.

(44) A square board is hung flat against a wall by means of a string attached to the extremities of its upper edge and passing round a smooth nail. Prove that if the length of the string is less than the diagonal of the board, there will be three positions of equilibrium.

(45) A point moves in a circle of radius r ft. with a uniform speed of $\pi r/6$ ft. per sec. Show that its mean acceleration during 6 seconds is $\pi r/18$ ft.-sec. units in a direction opposite to the initial direction of the velocity, and that the mean acceleration is $2/\pi$ times the magnitude of the uniform instantaneous acceleration.

(46) A particle is just supported by a rough inclined plane of variable inclination when its inclination is i. Find its acceleration up the plane when moving upwards on a line of greatest slope under the action of a force equal to twice its weight acting up the plane.

Ans. $g[2-\tan i(3\cos^2 i-\sin^2 i)]$.

(47) At a given instant a pendulum begins to oscillate in a vertical plane at a place of latitude 60°. Find after what time it will be apparently oscillating in a plane perpendicular to the former.

Ans. $1/2\sqrt{3}$ day.

(48) $ABCD$ is a square from which a corner AEF is cut off by a straight line drawn parallel to BD and at a distance from A equal to $\frac{3}{8}$ of the diagonal. Show that the distance of the centre of mass of AEF from A is $\frac{1}{4}$ of the diameter.

(49) A and B are points in a horizontal line. A uniform and smooth rod AC (weight $= W$) is fastened to a hinge at A and can swing in a vertical plane through AB. A string passes over a pulley at B, supporting at one end a body of weight P, and at the other being attached to a small smooth ring which slides on the rod. Prove that there will be equilibrium in any position if $W.AC=2P.AB$.

(50) If a conic section be described under the action of a force tending to a focus, the hodograph will be a circle.

(51) Show that 1 foot-grain is equivalent to 1·975 gramme-centimetres. [1 grain = 0·064799 gramme.]

(52) A rod (length $= l$) is fixed at one end about which it can move freely in any direction. When it is inclined to the horizon without motion at the angle α, a horizontal velocity V is communicated to its other end. Determine the velocity of the free end at the instant at which the rod becomes horizontal.

Ans. $(V^2 + 3lg \sin \alpha)^{\frac{1}{2}}$, inclined to the vertical at the angle $\tan^{-1}[V \cos \alpha / (V^2 \sin^2 \alpha + 3lg \sin \alpha)^{\frac{1}{2}}]$.

(53) A three-legged stool stands on the floor of an elevator sliding in its frame-work with perfect freedom. Show that it has four degrees of freedom.

(54) The distance of the centre of mass of half a hexagon inscribed in a circle from the centre is equal to $\frac{2r}{3\sqrt{3}}$ where r is the radius.

(55) Two uniform beams of given weight are in equilibrium in a vertical plane, the lower end of each beam resting on a horizontal floor and the upper ends being in contact. Show that the friction between either beam and the floor varies inversely as the sum of the tangents of the angles which the beams make with the floor.

(56) A particle moves in a parabola under the action of a constant force parallel to the axis. Show that the hodograph of its path is a straight line parallel to the axis.

(57) Show that one horse-power is equivalent to about 746 watts.

(58) A cone is revolving round its axis with a given angular velocity when the length of the axis begins to be diminished uniformly, and the vertical angle to be increased so that the volume of the cone remains unchanged. Show that if ω is the initial angular velocity of the cone, and h the initial length and r the rate of decrease of its axis, its angular velocity after any time t will be $\omega(1 - rt/h)$.

(59) Show that a body has two degrees of freedom, when two of its points are constrained to remain in given curves.

(60) A body consists of two portions and one of them is moved into a new position. Show that the line joining the two positions of the centre of mass of the whole is parallel to, and bears a fixed ratio to, the line joining the two positions of the centre of mass of the part moved.

(61) A regular hexagon is formed of rods jointed at their extremities. Strings are stretched between every pair of alternate angles of the hexagon so as to form two equilateral triangles. Show that the tension of any string is equal to $\frac{2}{3}$ of the sum of the tensions of the strings which cross it, minus $\frac{1}{3}$ of the tension of the string which is parallel to it.

(62) The kinetic energy of a particle, which is constrained to move in a circular path of radius r, varies as the square of its distance s, measured along the path from a fixed point in the path. Show that its tangential acceleration in any position is to its normal acceleration as $r : s$.

(63) If an agent working at the rate of one horse-power, perform the unit of work in the unit of time, and the acceleration of a falling body be unit of acceleration, a pound being the unit of mass, find the unit of (*a*) time and (*b*) length. [$g = 32$ ft.-sec. units.]

Ans. (*a*) $17\frac{3}{16}$ sec.; (*b*) $9453\frac{1}{8}$ ft.

(64) A rod is kept in a vertical position by means of two small rings and its lower end is supported on an inclined plane (inclination $= i$) which is freely moveable on a horizontal plane. Show that if v is the velocity of the rod and v' that of the inclined plane, $v = v' \tan i$.

(65) Show that if G be the centre of mass of the triangle ABC

$$3(GA^2 + GB^2 + GC^2) = AB^2 + BC^2 + CA^2.$$

(66) Two equal and similar rods AB, BC are freely hinged at B, and rest in a plane of greatest slope of a rough inclined plane, in a position of limiting equilibrium, with the end A hinged at a point in the plane, and the end C resting on the plane. If α, ϕ, ϵ are respectively the angle of inclination of the plane to the horizon, the angle of inclination of the rods to the plane, and the angle of friction, show that

$$3 \cos(\phi + \epsilon) \cos(\phi - \alpha) = \cos(\phi - \epsilon) \cos(\phi + \alpha).$$

(67) A point is moving in a straight line with an acceleration varying as its distance from a point in that line. Prove that the corresponding point in the hodograph moves with a similar acceleration.

(68) The mass of a railway train is 150 tons and the resistances to its motion (from air, friction, etc.) amount to 16 pounds-weight per ton. Find (a) the horse-power of the engine which can just keep it going at 60 miles an hour on a level plane, and (b) the greatest speed which an engine working at 200 horse-power can give it on a level plane.

Ans. (a) 384, (b) $31\frac{1}{4}$ miles per hour.

(69) A uniform rod (length $=2c$) moves in a vertical plane within a hemisphere with angular velocity ω. Show that if θ be the inclination of the rod to the horizon at any instant the horizontal and vertical velocities of its middle point have the magnitudes $c\omega \cos\theta$ and $c\omega \sin\theta$.

(70) The corners of a pyramid are cut off by planes parallel to the opposite faces. Show that if the portions cut off be of equal mass, the centre of mass of the remainder will coincide with that of the pyramid.

(71) Two uniform rods AB, AC of lengths a, b respectively, are of the same material and thickness and are smoothly jointed at A. A rigid weightless rod of length l is jointed at B to AB, and its other end D is fastened to a smooth ring sliding on AC. The system is hung over a smooth peg at A. Show that AC makes with the vertical an angle $\tan^{-1}[al/(b^2+a\sqrt{a^2-l^2})]$.

(72) If each unit involved in the measurement of g become m times its former value, show that the new value of g will be m times its former value also.

(73) A particle of 10 lbs. mass, whose motion is simple harmonic, has velocities 20 and 25 ft. per sec. at distances 10 and 8 ft. per sec. respectively from the centre of force. Find the work done during the motion from the distance 10 to the distance 8 feet.

Ans. 112·5 foot-poundals.

(74) A uniform rod in falling strikes, when in a horizontal position, with one end against a stone. Show that the impulse of the blow it receives is half that of the impulse of each of the blows which it would have received had both ends struck simultaneously against two stones, the blows being in all cases supposed to be at right angles to the rod.

(75) Show that a force of 100 dynes is equivalent to the weight of $1 \cdot 019 \times 10^{-4}$ kilogrammes.

(76) Show that in the direct impact of elastic balls of masses m and M and initial velocities v and V, and with coefficient of restitution e, an amount of kinetic energy equal to $(1-e)^2 \dfrac{Mm}{2(M+m)}(V-v)^2$ is lost.

(77) How much water will be pumped from a vertical cylindrical shaft of 10 feet diameter by an engine working for 6 hours at 200 horse-power, the water being discharged at a point 10 feet above the mouth of the shaft, and the surface of the water being initially 20 feet below the mouth of the shaft. [Density of water = 1,000 oz. per cub. ft.]

Ans. 2,157·1... tons.

(78) Determine the unit of time in order that with the foot as unit of length g (32 ft.-per-sec. per sec.) may have the value unity.

Ans. $1/4\sqrt{2}$ second.

(79) Find the work done on a body of 12 lbs. mass in falling to the earth's surface from a point 1,000 miles above it. [Earth's radius = 4,000 miles; $g = 32$ ft.-sec. units.]

Ans. 22,628·5... ft.-tons.

(80) A ball rolling on a horizontal plane strikes obliquely an equal ball at rest. The direction of motion of each ball after impact makes the same angle θ with that of the striking ball before impact. Show that the coefficient of restitution is equal to $\tan^2\theta$.

(81) If the weight of one ounce be the unit of force, one second the unit of time, and 162 the density in pounds per cubic foot of the standard substance, find the unit of length, g being taken to be 32 ft.-sec. units.

Ans. 4 inches.

(82) If from any point in the plane of a polygon perpendiculars be drawn to its sides, and if forces act along these perpendiculars, either all inwards or all outwards, each force being proportional to the side to which it is perpendicular, the system is in equilibrium.

(83) A rough heavy body bounded by a curved surface rests upon two others, which themselves rest upon a rough horizontal plane. Show that the three centres of mass and the four points of contact lie in one plane.

(84) Two points move in concentric circles of radii r and r'. When their radii vectores from the common centre are inclined θ radians, their angular velocities about the centre are ω and ω' respectively. Find the magnitude of their relative velocity.

Ans. $(\omega r^2 + \omega' r'^2 - 2\omega\omega' rr' \cos\theta)^{\frac{1}{2}}$.

(85) The ram of a pile-driver has a mass M, and a vertical fall h before reaching the pile. The pile has a mass M/n and is driven by one stroke through a vertical distance h/n. Find the mean resistance assuming that there is no recoil and that all work is expended in forcing the pile through the ground.

Ans. $Mg(n^2+n+1)/n$.

(86) Two equal balls of radius a are in contact and are struck simultaneously by a ball of radius c moving in the direction of their common tangent. All the balls are of the same material, the coefficient of restitution being e. Prove that the impinging ball will be reduced to rest if $2e = c^2(a+c)^2/(2a^4+a^3c)$.

(87) If two forces acting on a particle be represented by m times the line OA and n times the line OB, respectively, their resultant will be represented by $m+n$ times the line OC, C being the point on the line AB between A and B such that $m \,.\, AC = n \,.\, BC$.

(88) Prove that the centre of three parallel forces acting at the angular points of a triangle and proportional respectively to the opposite sides is at the centre of the inscribed circle when the forces are codirectional and at the centre of one or other of the escribed circles when they are not.

(89) Prove that the attractions of homogeneous spheres of different densities on a particle placed at the same distance from the centre of each, are as the products of the cubes of the radii and the densities of the spheres.

(90) The axle of a wheel (radius $=r$) is moving parallel to itself in one plane with a velocity v, and the wheel is turning about its axle with an angular velocity ω. Find the magnitude of the velocity of the end of a spoke whose inclination to the direction of the axle's motion is θ.

Ans. $(v^2+\omega^2r^2+2v\omega r\sin\theta)^{\frac{1}{2}}$.

(91) A particle of 10 lbs. mass moves with a simple harmonic motion of 2 inches amplitude and 0·04 seconds periodic time. Find (a) the potential energy at the extremity of its swing and (b) the kinetic energy at a distance of 1 inch from the mean position.

Ans. (a) 3426·9... ft.-pdls.; (b) 2570·2... ft.-pdls.

(92) An arc of a parabola is cut off by the double ordinate through the focus. Two bodies attached to the extremities of the arc sustain it with the axis inclined to the vertical at the angle $\pi/4$. The vertex being the point of suspension, show that the weight of one of the bodies is 3 times that of the other.

(93) Forces P and Q act on a particle, and their resultant is R. If any transversal cut their directions in the points L, M, N, respectively, show that $P/OL+Q/OM=R/ON$.

(94) If a rigid body be acted upon by four forces represented by the sides of a quadrilateral figure the axis of the resultant couple is proportional to its area.

(95) In a system of smooth pulleys such as that of Fig. 2, p. 426, if there are n moveable pulleys whose weights in order from the lowest are w_1, w_2, w_3, etc., and if W is the weight of the body which can be supported by a force F, and if the weight of the ropes be neglected,

$$2^nF=W+w_1+2w_2+2^2w_3+\text{etc.}+2^{n-1}w_n.$$

(96) Two particles start together from rest and move in directions perpendicular to one another. One moves uniformly with a

velocity of 3 ft. per sec., the other under the action of a constant force of 20 poundals. Determine the mass of the particle on which this force acts, if the particles at the end of 4 seconds are 20 feet apart.

Ans. 10 lbs.

(97) A uniform spherical shell of gravitating matter has an internal diameter of 4 feet, an external diameter of 6 feet and a density of 4 lbs. per cubic foot. Find in lb.-ft. units the potential at a point distant 10 ft. from the centre of the shell.

Ans. $10\cdot 1\dot{3}\pi$.

(98) In a false balance a body of weight P appears to weigh Q, and one of weight P' to weigh Q'. Prove that the real weight X of what appears to weigh Y is given by the equation

$$X(Q-Q')=Y(P-P')+P'Q-PQ'.$$

(99) $ABCDEF$ is a regular hexagon, and at A forces act represented by AB, $2AC$, $3AD$, $4AE$, $5AF$. Show that the length of the line representing their resultant is $\sqrt{351}\,.\,AB$.

(100) Two forces P and Q act upon a body along two given straight lines. Prove that

$$1/P=(\cos\theta)/R+(a\sin\theta)/G,$$

$$1/Q=(\cos\phi)/R+(b\sin\phi)/G,$$

θ and ϕ being the angles made by the given straight lines with the central axis, a and b the shortest distance between these lines and the central axis, R the resultant force and G the resultant couple.

(101) Show that a stress of 40 grammes-weight per square centimetre is equivalent to one of 0·5689... pound's-weight per sq. inch.

(102) If s and s' are the spaces traversed by a point moving with uniform acceleration in a straight line in the times t and t' respectively, reckoned from the same instant, show that the acceleration and the initial velocity are, respectively,

$$\frac{2(s't-st')}{tt'(t'-t)} \text{ and } \frac{st'^2-s't^2}{tt'(t'-t)}.$$

(103) A gun is suspended freely by two equal parallel cords and a shot is fired from it. Prove that the range on a horizontal plane is, for a given gun and shot, directly proportional both to the height through which the gun rises in the recoil and to the tangent of its initial inclination to the horizon.

(104) Find the force exerted by two equal uniform discs (radius $=a$, distance $=c$, surface density $=\rho$) placed perpendicularly to the line joining their centres, on a particle of unit mass in that line at a distance b from the nearer disc, it being given that the one disc attracts, while the other repels, according to the gravitational law.

Ans. $2\pi\rho[(b+c)/\sqrt{a^2+(b+c)^2}-b/\sqrt{a^2+b^2}]$.

(105) Find the moment of inertia of a fly-wheel (mass $=M$) formed by cutting from a circular plate of radius r_1 a circular portion (concentric with the plate) of radius r_2.

Ans. $\frac{1}{2}M(r_1^2+r_2^2)$.

(106) A hollow vessel has the form of a pyramid, four of whose five faces are equilateral triangles (side $=a$). It is placed with its square face on a horizontal plane and filled with a liquid of density ρ through a small aperture in the vertex. Find the integral stress on the four triangular faces.

Ans. $a^3\rho g\sqrt{2/3}$.

(107) Two inclined planes intersect in a horizontal line, their inclinations to the horizon being α and β. If a particle be projected at right angles to the former from a point in it so as to strike the other at right angles, the velocity of projection must be

$$\sin\beta[2ga/(\sin\alpha-\sin\beta\cos(\alpha+\beta))]^{\frac{1}{2}},$$

a being the distance of the point of projection from the intersection of the planes.

(108) Two particles, of masses 9,820 and 1,964 grammes respectively, attract one another. Find the acceleration of either relative to the other, when the distance between them is 4 cm.

Ans. 0·75 cm.-sec. units.

(109) Find the acceleration produced by a mass of 1 kilogramme in a particle at a distance of 1 metre. [Earth's mass $=6{\cdot}14\times10^{27}$ grammes ; earth's radius $=6{\cdot}37\times10^{8}$ cm.; $g=981$ cm.-sec. units.]

Ans. $6{\cdot}48\times10^{-9}$.

(110) A moment of inertia is expressed in terms of the units of the ft.-lb.-sec. gravitational system. By what number must its value be multiplied that it may be expressed in terms of the metre-kilogramme-second gravitational system.

Ans. 0·138....

(111) Two particles are projected from two given points in the same vertical line with the same velocities. Prove that lines touching the path of the lower will cut off from the path of the upper, arcs described in equal times.

(112) One bullet is fired towards another bullet which is let fall at the same instant. Prove that if, on meeting (see 119, Ex. 4), they coalesce, the *latus rectum* of their joint path will be one-fourth of that of the original path of the first bullet.

(113) A uniform bar of length a rests suspended by two strings of lengths l and l' fastened to the ends of the bar and to two fixed points in the same horizontal line at a distance c apart. Prove that if the directions of the strings are perpendicular the ratio of their tensions is $al+cl' : al'+cl$.

(114) In the expression for the attraction of two particles, $F=kmm'/d^2$, how does the value of k depend upon the units of mass, length, and time?

Ans. Its dimensions are $[M]^{-1}[L]^3[T]^{-2}$.

(115) Show that the radius of gyration of a uniform square disc (side $=a$) about one of its diagonals is $a/\sqrt{12}$.

(116) A particle describes an ellipse under a force directed towards its centre. Show that the time between the extremities of conjugate diameters will be constant.

(117) A particle is dropped from a point A and a second equal particle is simultaneously projected vertically upwards from a point B so that the balls impinge, the stress during impact being in a line inclined $\tan^{-1}\sqrt{e}$ (e being the coefficient of restitution) to the vertical. Prove that both balls will strike the horizontal plane through B simultaneously, and that if the velocity of projection at B be that due to AB (*i.e.*, that which the particle would have had had it fallen through the vertical distance AB), their distance there will be $AB\sqrt{3e}$.

(118) If a body, attached at its centre of mass to one end of a string of length r, the other end being attached to a fixed point in a smooth horizontal plane, make n revolutions in one unit of time, prove that the ratio of the tension in the string to the force exerted on the plane is $4\pi^2 n^2 r : g$.

(119) Find the time of a small double oscillation under gravity of a uniform one foot cube suspended by one edge as horizontal axis.

Ans. 1·07... second.

(120) A particle describes a parabolic orbit under a force directed towards the focus. Show that the sum of the squares of the velocities at the extremities of a focal chord is constant.

(121) A body of mass P pulls one of mass Q over a smooth pulley, and Q in ascending, as it passes a certain point A, catches and carries with it a third body B, which in its descent is again deposited at A. Supposing no jerk to occur when B is caught up and that Q oscillates through equal distances above and below A, prove that the mass of B is $(P^2 - Q^2)/Q$.

(122) A uniform triangular lamina suspended from a fixed point by three cords attached to its three vertices is in equilibrium. Show that the tensions in the cords are proportional to their lengths.

(123) Three inches of rain fell in a certain district ($g = 32$ ft.-sec. units) in 12 hours. Assuming that the drops fell from a height of a quarter of a mile and neglecting the resistance of the air, find the pressure on the ground due to the rain during the storm. [The mass of a cubic foot of rain water = 1,000 oz.]

Ans. 0·105... poundals per sq. foot.

(124) Show that in the case of a right-angled isosceles triangular plate the times of small oscillations are the same about horizontal axes perpendicular to its plane through its vertex and through the middle point of its base.

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